The Remarkable Limit: $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$, x is in radians.

① The sine function is an odd function so $\sin(-x) = -\sin x$

If $x > 0$,

$$\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$$

$\therefore \frac{\sin \square}{\square}$ is the same whether $\square > 0$ or $\square < 0$

② That means we only need to consider $x > 0$ as $x \to 0$.

We are saying that the

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x}$$

i.e. limit as x approaches zero from the left is equal to the limit as x approaches zero

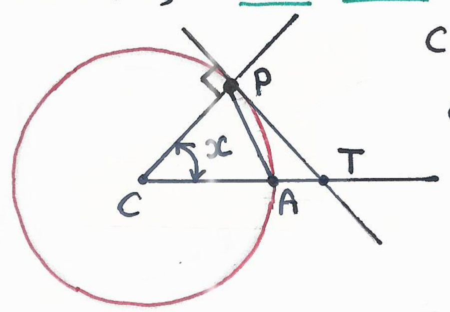
③ from the right, if either exists.

That means it's sufficient to look at $x > 0$ as $x \to 0$.

Since $x \to 0$, we can claim $0 < x < \frac{\pi}{2}$

(x is a small positive number)

④ Consider a diagram with circle of radius 1, a unit circle.



$CP = CA = 1$

and

$TP \perp CP$ (line through P and T is a tangent line)

⑤ Line segment PT is tangent to the circle at point P.

We can compare three areas: those of $\triangle CAP$, of sector OAP (part of circle) and of $\triangle CTP$.

$$A_{\triangle CAP} < A_{\text{sector } CAP} < A_{\triangle CTP}$$

⑤

$$\frac{1}{2}(CA)(CP\sin x) < \frac{x}{2} < \frac{1}{2}(CP)(PT)$$

Remark: to find area of a sector of x radians:

$$\frac{\pi r^2}{2\pi} \cdot x = \frac{r^2}{2} x = A_{\text{sector}}$$

since $r = 1$, $A = \frac{(1)^2 x}{2} = \frac{x}{2}$

⑥

$$\frac{1}{2}(1)(\sin x) < \frac{x}{2} < \frac{1}{2} PT(1)$$

From $\triangle CTP$: $\frac{PT}{CP} = \tan x$

$\therefore PT = \tan x$, as $CP = 1$

$$\therefore \frac{\sin x}{2} < \frac{x}{2} < \frac{1}{2}\tan x$$

Since $\sin x > 0$, $0 < x < \frac{\pi}{2}$.

we multiply through by 2 and divide

⑦ through by $\sin x$.

$$\sin x < x < \tan x$$

$$\sin x < x < \frac{\sin x}{\cos x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \text{or} \quad \cos x < \frac{\sin x}{x} < 1$$

The function $y = \cos x$ is continuous and $\cos 0 = 1$. As $x \to 0$, $\frac{\sin x}{x}$ squeezes from both sides of inequality to 1.

$$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$$