Investigate 2 How is symmetry represented in the equation of a polynomial function?

Tools

- graphing calculator

Optional

- computer with *The Geometer's Sketchpad®*

A: *Even-Degree Polynomial Functions*

1. a) Graph each function on a separate set of axes. Sketch each graph in your notebook.

i) $f(x) = x^4$

ii) $f(x) = x^4 - 8x^2$

iii) $f(x) = -x^4 + 9x^2$

iv) $f(x) = -x^6 + 7x^4 + 3x^2$

b) Reflect What type of symmetry does each graph have?

c) Reflect How can the exponents of the terms be used to identify these as even functions?

2. a) For each function in step 1a), determine $f(-x)$ by substituting $-x$ for x.

b) Reflect Compare $f(x)$ and $f(-x)$. What do you notice?

B: *Odd-Degree Polynomial Functions*

1. a) Graph each function on a separate set of axes. Sketch each graph in your notebook.

i) $f(x) = x^3$

ii) $f(x) = x^3 - 4x$

iii) $f(x) = -x^5 + 16x^3$

iv) $f(x) = -x^5 + 5x^3 + 6x$

b) Reflect What type of symmetry does each graph have?

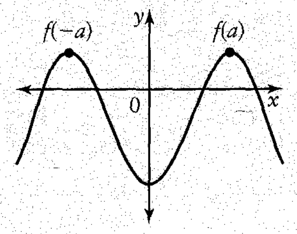
c) Reflect How can the exponents of the terms be used to identify these as odd functions?

2. a) For each function in step 1a), determine $f(-x)$ and $-f(x)$.

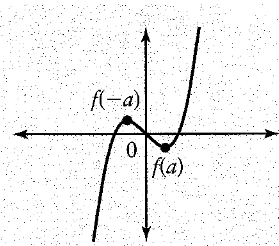
b) Reflect Compare $-f(x)$ and $f(-x)$. What do you notice?

3. Reflect Use the results of parts A and B to describe two ways that the equation of a polynomial function can be used to determine the type of symmetry exhibited by the graph of that function.

An even-degree polynomial function is an **even function** if the exponent of each term of the equation is even. An even function satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$. An even function is symmetric about the y-axis.



An odd-degree polynomial function is an **odd function** if each term of the equation has an odd exponent. An odd function satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$. An odd function is rotationally symmetric about the origin.



Example 3 Identify Symmetry

Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Verify your response.

a) $f(x) = 2x^4 - 5x^2 + 4$

b) $f(x) = -3x^5 + 9x^3 + 2x$

c) $f(x) = 2x(x + 1)(x - 2)$

d) $f(x) = x^6 - 4x^3 + 6x^2 - 4$

Solution

a) Since the exponent of each term is even, $f(x) = 2x^4 - 5x^2 + 4$ is an even function and has line symmetry about the y-axis.
Verify that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= 2(-x)^4 - 5(-x)^2 + 4 && \text{Substitute } -x \text{ in the equation.} \\ &= 2x^4 - 5x^2 + 4 \\ &= f(x) \end{aligned}$$

b) Since the exponent of each term is odd, $f(x) = -3x^5 + 9x^3 + 2x$ is an odd function and has point symmetry about the origin. Verify that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= -3(-x)^5 + 9(-x)^3 + 2(-x) && \text{Substitute } -x. \\ &= 3x^5 - 9x^3 - 2x \end{aligned}$$

$$\begin{aligned} -f(x) &= -(-3x^5 + 9x^3 + 2x) && \text{Multiply } f(x) \text{ by } -1. \\ &= 3x^5 - 9x^3 - 2x \end{aligned}$$

The resulting expressions are equal, so $f(-x) = -f(x)$.

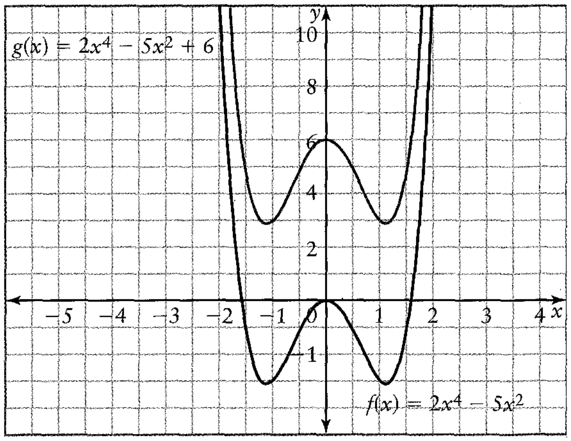
c) Since $f(x) = 2x(x + 1)(x - 2)$ is a cubic function, it may be odd and thus have point symmetry about the origin.

$$\begin{aligned} f(-x) &= 2(-x)(-x + 1)(-x - 2) && \text{Substitute } -x. \\ &= -2x(-1)(x - 1)(-1)(x + 2) && \text{Factor } -1 \text{ from each factor.} \\ &= -2x(x - 1)(x + 2) \end{aligned}$$

$$-f(x) = -2x(x + 1)(x - 2) \qquad \text{Multiply } f(x) \text{ by } -1.$$

The resulting expressions are not equal, so the function is not odd and does not have point symmetry about the origin.

d) Some exponents in $f(x) = x^6 - 4x^3 + 6x^2 - 4$ are even and some are odd, so the function is neither even nor odd and does not have line symmetry about the y-axis or point symmetry about the origin.



When a constant term is added to an even function, the function remains even. For example, the graph of $g(x) = 2x^4 - 5x^2 + 6$ represents a vertical translation of 6 units up of the graph of $f(x) = 2x^4 - 5x^2$. Thus, since $f(x) = 2x^4 - 5x^2$ is even and has line symmetry, the same is true for $g(x) = 2x^4 - 5x^2 + 6$.

CONNECTIONS

Recall that constant terms can be thought of as coefficients of x^0.

KEY CONCEPTS

- The graph of a polynomial function can be sketched using the x-intercepts, the degree of the function, and the sign of the leading coefficient.
- The x-intercepts of the graph of a polynomial function are the roots of the corresponding polynomial equation.
- When a polynomial function is in factored form, the zeros can be easily determined from the factors. When a factor is repeated n times, the corresponding zero has order n.
- The graph of a polynomial function changes sign only at x-intercepts that correspond to zeros of odd order. At x-intercepts that correspond to zeros of even order, the graph touches but does not cross the x-axis.
- An even function satisfies the property $f(-x) = f(x)$ for all x in its domain and is symmetric about the y-axis. An even-degree polynomial function is an even function if the exponent of each term is even.
- An odd function satisfies the property $f(-x) = -f(x)$ for all x in its domain and is rotationally symmetric about the origin. An odd-degree polynomial function is an odd function if the exponent of each term is odd.

Communicate Your Understanding

C1 Are all even-degree polynomial functions even? Are all odd-degree polynomial functions odd? Explain.

C2 Why is it useful to express a polynomial function in factored form?

C3 **a)** What is the connection between the order of the zeros of a polynomial function and the graph?

b) How can you tell from a graph if the order of a zero is 1, 2, or 3?

C4 How can symmetry be used to sketch a graph of a polynomial function?