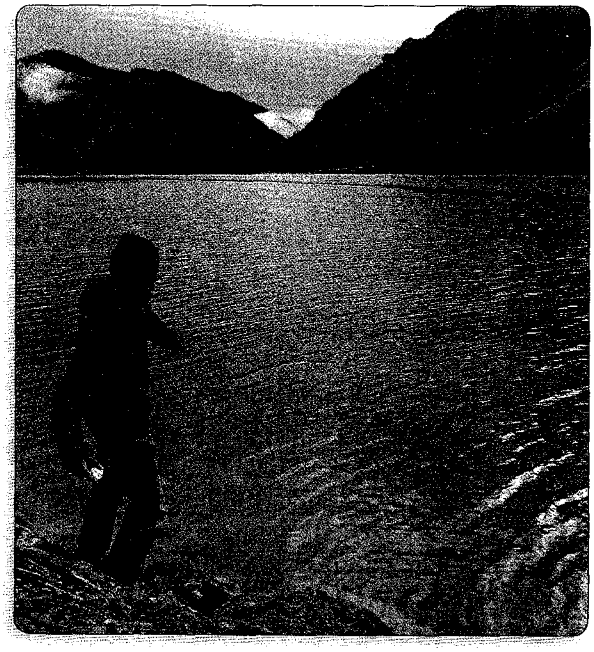
1.1 Power Functions



A rock that is tossed into the water of a calm lake creates ripples that move outward in a circular pattern. The area, A, spanned by the ripples can be modelled by the function $A(r) = \pi r^2$, where r is the radius. The volume, V, of helium in a spherical balloon can be modelled by the function $V(r) = \frac{4}{3}\pi r^3$, where r is again the radius. The functions that represent each situation are called **power functions**. A power function is the simplest type of **polynomial function** and has the form $f(x) = ax^n$, where x is a variable, a is a real number, and n is a whole number.

CONNECTIONS

Polynomials are the building blocks of algebra. Polynomial functions can be used to create a variety of other types of functions and are important in many areas of mathematics, including calculus and numerical analysis. Outside mathematics, the basic equations in economics and many physical sciences are polynomial equations.

A polynomial expression is an expression of the form

$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$,

where

- n is a whole number
- x is a variable
- the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers
- the **degree** of the function is n, the exponent of the greatest power of x
- a_n, the coefficient of the greatest power of x, is the **leading coefficient**
- a_0, the term without a variable, is the **constant term**

A **polynomial function** has the form

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$

Polynomial functions are typically written in descending order of powers of x. The exponents in a polynomial do not need to decrease consecutively; that is, some terms may have zero as a coefficient. For example, $f(x) = 4x^3 + 2x - 1$ is still a polynomial function even though there is no x^2-term. A constant function, of the form $f(x) = a_0$, is also a type of polynomial function (where $n = 0$), as you can write the constant term a_0 in the form a_0x^0.

Investigate What are the key features of the graphs of power functions?

Tools
- graphing calculator

1. Match each graph with the corresponding function. Justify your choices. Use a graphing calculator if necessary.

a) $y = x$

b) $y = x^2$

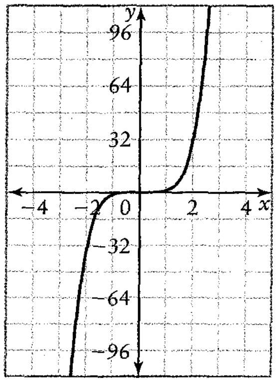
c) $y = x^3$

d) $y = x^4$

e) $y = x^5$

f) $y = x^6$

i)



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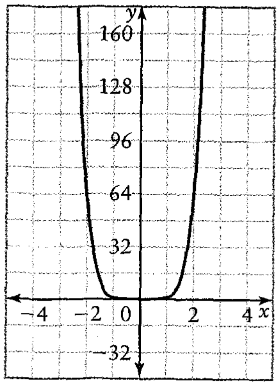

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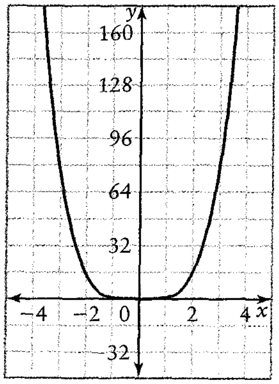

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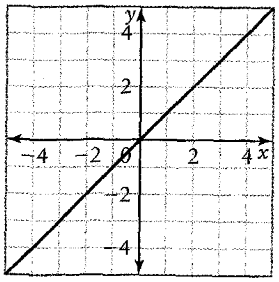

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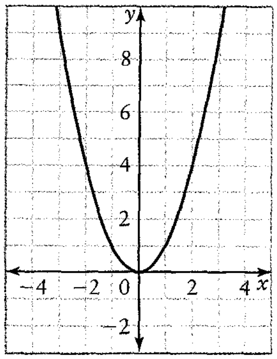

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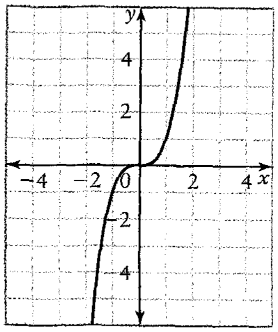


CONNECTIONS

Some power functions have special names that are associated with their degree.

Power Function	Degree	Name
$y = a$	0	constant
$y = ax$	1	linear
$y = ax^2$	2	quadratic
$y = ax^3$	3	cubic
$y = ax^4$	4	quartic
$y = ax^5$	5	quintic

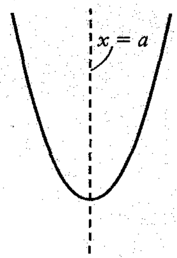
CONNECTIONS

Recall that a relation is a function if for every x-value there is only one y-value. The graph of a relation represents a function if it passes the vertical line test, that is, if a vertical line drawn anywhere along the graph intersects that graph at no more than one point.

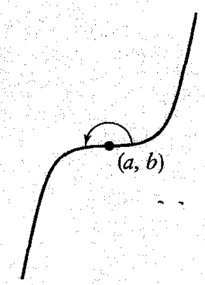
The **end behaviour** of the graph of a function is the behaviour of the y-values as x increases (that is, as x approaches positive infinity, written as $x \to \infty$) and as x decreases (that is, as x approaches negative infinity, written as $x \to -\infty$).

CONNECTIONS

- A graph has **line symmetry** if there is a line $x = a$ that divides the graph into two parts such that each part is a reflection of the other in the line $x = a$.



- A graph has **point symmetry** about a point (a, b) if each part of the graph on one side of (a, b) can be rotated 180° to coincide with part of the graph on the other side of (a, b).



2. a) Reflect Decide whether each graph in step 1 represent a linear, a quadratic, a cubic, a quartic, or a quintic function. Justify your answer.

b) Reflect Explain why each graph in step 1 represents a function.

3. a) State these key features for each graph:

i) the domain

ii) the range

iii) the intercepts

b) Describe the end behaviour of each graph as

i) $x \to \infty$

ii) $x \to -\infty$

4. a) Which graphs have both ends extending upward in quadrants 1 and 2 (that is, start high and end high)?

b) Decide whether each graph has line symmetry or point symmetry. Explain.

c) Reflect Describe how the graphs are similar. How are the equations similar?

5. a) Which graphs have one end extending downward in quadrant 3 (start low) and the other end extending upward in quadrant 1 (end high)?

b) Decide whether each graph has line symmetry or point symmetry. Explain.

c) Reflect Describe how the graphs are similar. How are the equations similar?

6. Reflect Summarize your findings for each group of power functions in a table like this one.

Key Features of the Graph	$y = x^n$, n is odd	$y = x^n$, n is even
Domain		
Range		
Symmetry		
End Behaviour		

7. a) Graph the function $y = x^n$ for $n = 2, 4$, and 6.

b) Describe the similarities and differences between the graphs.

c) Reflect Predict what will happen to the graph of $y = x^n$ for larger even values of n.

d) Check your prediction in part c) by graphing two other functions of this form.

8. a) Graph the function $y = x^n$ for $n = 1, 3$, and 5.

b) Describe the similarities and differences between the graphs.

c) Reflect Predict what will happen to the graph of $y = x^n$ for larger odd values of n.

d) Check your prediction in part c) by graphing two more functions of this form.

Example 1 Recognize Polynomial Functions

Determine which functions are polynomials. Justify your answer. State the degree and the leading coefficient of each polynomial function.

a) $g(x) = \sin x$

b) $f(x) = 2x^4$

c) $y = x^3 - 5x^2 + 6x - 8$

d) $g(x) = 3^x$

Solution

a) $g(x) = \sin x$

This a trigonometric function, not a polynomial function.

b) $f(x) = -2x^4$

This is a polynomial function of degree 4. The leading coefficient is -2.

c) $y = x^3 - 5x^2 + 6x - 8$

This is a polynomial function of degree 3. The leading coefficient is 1.

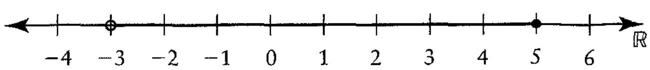
d) $g(x) = 3^x$

This is not a polynomial function but an exponential function, since the base is a number and the exponent is a variable.

Interval Notation

In this course, you will often describe the features of the graphs of a variety of types of functions in relation to real-number values. Sets of real numbers may be described in a variety of ways:

- as an inequality, $-3 < x \leq 5$
- in interval (or bracket) notation $(-3, 5]$



- graphically, on a number line

Intervals that are infinite are expressed using the symbol ∞ (infinity) or $-\infty$ (negative infinity).

Square brackets indicate that the end value is included in the interval, and round brackets indicate that the end value is not included.

A round bracket is used at infinity since the symbol ∞ means "without bound."

Below is a summary of all possible intervals for real numbers a and b, where $a < b$.

Bracket Interval	Inequality	Number Line	In Words
			The set of all real numbers x such that
(a, b)	$a < x < b$	a b $\mathbb{R}$	x is greater than a and less than b
$(a, b]$	$a < x \leq b$	a b $\mathbb{R}$	x is greater than a and less than or equal to b
$[a, b)$	$a \leq x < b$	a b $\mathbb{R}$	x is greater than or equal to a and less than b
$[a, b]$	$a \leq x \leq b$	a b $\mathbb{R}$	x is greater than or equal to a and less than or equal to b
$[a, \infty)$	$x \geq a$	a $\mathbb{R}$	x is greater than or equal to a
$(-\infty, a]$	$x \leq a$	a $\mathbb{R}$	x is less than or equal to a
(a, ∞)	$x > a$	a $\mathbb{R}$	x is greater than a
$(-\infty, a)$	$x < a$	a $\mathbb{R}$	x is less than a
$(-\infty, \infty)$	$-\infty < x < \infty$	$\mathbb{R}$	x is an element of the real numbers

Example 2 Connect the Equations and Features of the Graphs of Power Functions

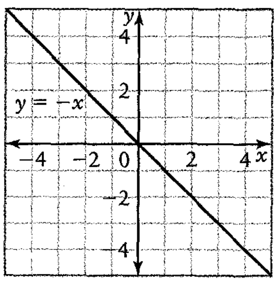
For each function

i) state the domain and range

ii) describe the end behaviour

iii) identify any symmetry

a)



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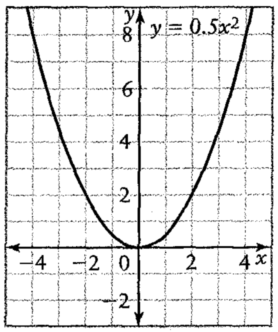

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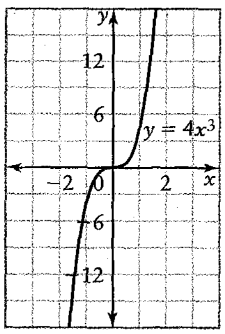


Solution

a) $y = -x$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}\}$ or $(-\infty, \infty)$

ii) The graph extends from quadrant 4 to quadrant 2.
Thus, as $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to -\infty$.

iii) The graph has point symmetry about the origin $(0, 0)$.

b) $y = 0.5x^2$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}, y \geq 0\}$ or $[0, \infty)$

ii) The graph extends from quadrant 2 to quadrant 1.
Thus, as $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to \infty$.

iii) The graph has line symmetry in the y-axis.

c) $y = 4x^3$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}\}$ or $(-\infty, \infty)$

ii) The graph extends from quadrant 3 to quadrant 1.
Thus, as $x \to -\infty$, $y \to -\infty$, and as $x \to \infty$, $y \to \infty$.

iii) The graph has point symmetry about the origin.

Example 3 Describe the End Behaviour of Power Functions

Write each function in the appropriate row of the second column of the table. Give reasons for your choices.

$y = 2x$ $\quad y = 5x^6$ $\quad y = -3x^2$ $\quad y = x^7$

$y = -\frac{2}{5}x^9$ $\quad y = -4x^5$ $\quad y = x^{10}$ $\quad y = -0.5x^8$

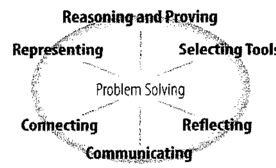
End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1		
Extends from quadrant 2 to quadrant 4		
Extends from quadrant 2 to quadrant 1		
Extends from quadrant 3 to quadrant 4		

Solution

End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1	$y = 2x, y = x^7$	odd exponent, positive coefficient
Extends from quadrant 2 to quadrant 4	$y = -\frac{2}{5}x^9, y = -4x^5$	odd exponent, negative coefficient
Extends from quadrant 2 to quadrant 1	$y = 5x^6, y = x^{10}$	even exponent, positive coefficient
Extends from quadrant 3 to quadrant 4	$y = -3x^2, y = -0.5x^8$	even exponent, negative coefficient

Example 4 Connecting Power Functions and Volume

Helium is pumped into a large spherical balloon designed to advertise a new product. The volume, V, in cubic metres, of helium in the balloon is given by the function $V(r) = \frac{4}{3}\pi r^3$, where r is the radius of the balloon, in metres, and $r \in [0, 5]$.



a) Graph $V(r)$.

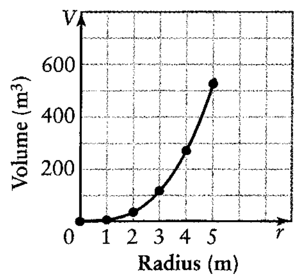
b) State the domain and range in this situation.

c) Describe the similarities and differences between the graph of $V(r)$ and the graph of $y = x^3$.

Solution

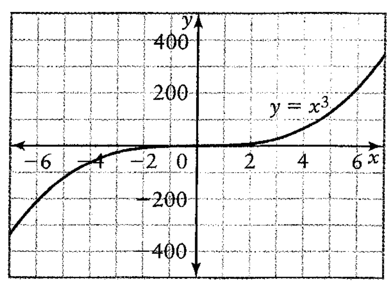
a) Make a table of values, plot the points, and connect them using a smooth curve.

r (m)	$V(r) = \frac{4}{3}\pi r^3$ (m^3)
0	$\frac{4}{3}\pi(0)^3 = 0$
1	$\frac{4}{3}\pi(1)^3 \doteq 4.2$
2	$\frac{4}{3}\pi(2)^3 \doteq 33.5$
3	$\frac{4}{3}\pi(3)^3 \doteq 113.1$
4	$\frac{4}{3}\pi(4)^3 \doteq 268.1$
5	$\frac{4}{3}\pi(5)^3 \doteq 523.6$



b) The domain is $r \in [0, 5]$. The range is approximately $V \in [0, 523.6]$.

c) The graph of $y = x^3$ is shown.



Similarities: The functions $V(r) = \frac{4}{3}\pi r^3$ and $y = x^3$ are both cubic, with positive leading coefficients. Both graphs pass through the origin (0, 0) and have one end that extends upward in quadrant 1.

Differences: the graph of $V(r)$ has a restricted domain. Since the two functions are both cubic power functions that have different leading coefficients, all points on each graph, other than (0, 0), are different.

KEY CONCEPTS

- A polynomial expression has the form
  $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$
  where
  - n is a whole number
  - x is a variable
  - the coefficients $a_0, a_1, \ldots, a_n$ are real numbers
  - the degree of the function is n, the exponent of the greatest power of x
  - a_n, the coefficient of the greatest power of x, is the leading coefficient
  - a_0, the term without a variable, is the constant term
- A polynomial function has the form
  $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$
- A power function is a polynomial of the form $y = ax^n$, where n is a whole number.
- Power functions have similar characteristics depending on whether their degree is even or odd.
- Even-degree power functions have line symmetry in the y-axis, $x = 0$.
- Odd-degree power functions have point symmetry about the origin, (0, 0).

Communicate Your Understanding

C1 Explain why the function $y = 3$ is a polynomial function.

C2 How can you use a graph to tell whether the leading coefficient of a power function is positive or negative?

C3 How can you use a graph to tell whether the degree of a power function is even or odd?

C4 State a possible equation for a power function whose graph extends

a) from quadrant 3 to quadrant 1

b) from quadrant 2 to quadrant 1

c) from quadrant 2 to quadrant 4

d) from quadrant 3 to quadrant 4

Practise

For help with questions 1 and 2, refer to Example 1.

1. Identify whether each is a polynomial function. Justify your answer.

a) $p(x) = \cos x$ **b)** $h(x) = -7x$

c) $f(x) = 2x^4$ **d)** $y = 3x^5 - 2x^3 + x^2 - 1$

e) $k(x) = 8^x$ **f)** $y = x^{-3}$

2. State the degree and the leading coefficient of each polynomial.

a) $y = 5x^4 - 3x^3 + 4$ **b)** $y = -x + 2$

c) $y = 8x^2$ **d)** $y = -\frac{x^3}{4} + 4x - 3$

e) $y = -5$ **f)** $y = x^2 - 3x$