The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

5th Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 10, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. **(A)** Six people are fundraising, so each must raise $\$1500/6 = \$250$.

2. **(B)** Because

$$\P(1,2,3) = \frac{1}{2-3} = -1, \quad \P(2,3,1) = \frac{2}{3-1} = 1, \text{ and}$$
$$\P(3,1,2) = \frac{3}{1-2} = -3,$$

we have

$$\P\left(\P(1,2,3), \P(2,3,1), \P(3,1,2)\right) = \P(-1,1,-3) = \frac{-1}{1-(-3)} = -\frac{1}{4}.$$

3. **(E)** Since \$20 is 2000 cents, she pays $(0.0145)(2000) = 29$ cents per hour in local taxes.

4. **(D)** The equation implies that either

$$x-1 = x-2 \quad \text{or} \quad x-1 = -(x-2)$$

The first equation has no solution, and the solution to the second equation is $x = 3/2$.

OR

Since $|x-a|$ is the distance of x from a, x must be equidistant from 1 and 2. Hence $x = 3/2$.

5. **(C)** The number of three-point sets that can be chosen from the nine grid points is

$$\binom{9}{3} = \frac{9!}{3! \cdot 6!} = 84.$$

Eight of these sets consist of three collinear points:
3 sets of points lie on vertical lines, 3 on horizontal lines, and 2 on diagonals. Hence the probability is $8/84 = 2/21$.

6. **(E)** Bertha has $30-6 = 24$ granddaughters, none of whom have any daughters. The granddaughters are the children of $24/6 = 4$ of Bertha's daughters, so the number of women having no daughters is $30-4 = 26$.

7. **(C)** There are five layers in the stack, and each of the top four layers has one less orange in its length and width than the layer on which it rests. Hence the total number of oranges in the stack is

$$5 \cdot 8 + 4 \cdot 7 + 3 \cdot 6 + 2 \cdot 5 + 1 \cdot 4 = 100.$$

8. **(B)** After three rounds the players A, B, and C have 14, 13, and 12 tokens, respectively. Every subsequent three rounds of play reduces each player's supply of tokens by one. After 36 rounds they have 3, 2, and 1 token, respectively, and after the 37^{th} round Player A has no tokens.

9. **(B)** Let x, y, and z be the areas of $\triangle ADE$, $\triangle BDC$, and $\triangle ABD$, respectively. The area of $\triangle ABE$ is $(1/2)(4)(8) = 16 = x + z$, and the area of $\triangle BAC$ is $(1/2)(4)(6) = 12 = y + z$. The requested difference is

$$x - y = (x + z) - (y + z) = 16 - 12 = 4.$$

10. **(D)** The result will occur when both A and B have either 0, 1, 2, or 3 heads, and these probabilities are shown in the table.

Heads	0	1	2	3
A	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
B	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$

The probability of both coins having the same number of heads is

$$\frac{1}{8} \cdot \frac{1}{16} + \frac{3}{8} \cdot \frac{4}{16} + \frac{3}{8} \cdot \frac{6}{16} + \frac{1}{8} \cdot \frac{4}{16} = \frac{35}{128}.$$

11. **(C)** Let r, h, and V, respectively, be the radius, height, and volume of the jar that is currently being used. The new jar will have a radius of $1.25r$ and volume V. Let H be the height of the new jar. Then

$$\pi r^2 h = V = \pi (1.25r)^2 H, \quad \text{so} \quad \frac{H}{h} = \frac{1}{(1.25)^2} = 0.64.$$

Thus H is 64% of h, so the height must be reduced by $(100 - 64)\% = 36\%$.

OR

Multiplying the diameter by $5/4$ multiplies the area of the base by $(5/4)^2 = 25/16$, so in order to keep the same volume, the height must be multiplied by $16/25$. Thus the height must be decreased by $9/25$, or 36%.

12. **(C)** A customer makes one of two choices for each condiment, to include it or not to include it. The choices are made independently, so there are $2^8 = 256$ possible combinations of condiments. For each of those combinations there are three choices regarding the number of meat patties, so there are altogether $(3)(256) = 768$ different kinds of hamburger.

13. **(D)** Because each man danced with exactly three women, there were $(12)(3) = 36$ pairs of men and women who danced together. Each woman had two partners, so the number of women who attended is $36/2 = 18$.

14. **(A)** If n is the number of coins in Paula's purse, then their total value is $20n$ cents. If she had one more quarter, she would have $n+1$ coins whose total value in cents could be expressed both as $20n + 25$ and as $21(n + 1)$. Therefore

$$20n + 25 = 21(n + 1), \quad \text{so} \quad n = 4.$$

Since Paula has four coins with a total value of 80 cents, she must have three quarters and one nickel, so the number of dimes is 0.

15. **(D)** Because

$$\frac{x+y}{x} = 1 + \frac{y}{x} \quad \text{and} \quad \frac{y}{x} < 0,$$

the value is maximized when $|y/x|$ is minimized, that is, when $|y|$ is minimized and $|x|$ is maximized. So $y = 2$ and $x = -4$ gives the largest value, which is $1 + (-1/2) = 1/2$.

16. **(D)** All of the squares of size 5×5, 4×4, and 3×3 contain the black square and there are

$$1^2 + 2^2 + 3^2 = 14$$

of these. In addition, 4 of the 2×2 squares and 1 of the 1×1 squares contain the black square, for a total of $14 + 4 + 1 = 19$.

17. **(C)** When they first meet, they have run a combined distance equal to half the length of the track. Between their first and second meetings, they run a combined distance equal to the full length of the track. Because Brenda runs at a constant speed and runs 100 meters before their first meeting, she runs $2(100) = 200$ meters between their first and second meetings. Therefore the length of the track is $200 + 150 = 350$ meters.

18. **(A)** The terms of the arithmetic progression are 9, $9 + d$, and $9 + 2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11 + d)^2 = 9(29 + 2d) \quad \text{so} \quad d^2 + 4d - 140 = 0.$$

Thus $d = 10$ or $d = -14$. The corresponding geometric progressions are 9, 21, 49 and 9, -3, 1, so the smallest possible value for the third term of the geometric progression is 1.

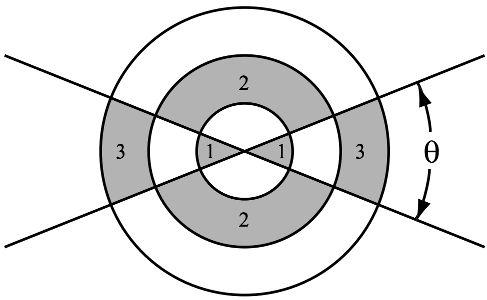
19. **(C)** If the stripe were cut from the silo and spread flat, it would form a parallelogram 3 feet wide and 80 feet high. So the area of the stripe is $3(80) = 240$ square feet.

20. **(D)** First, assume that $AB = 1$, and let $ED = DF = x$. By the Pythagorean Theorem $x^2 + x^2 = EF^2 = EB^2 = 1^2 + (1 - x)^2$, so $x^2 = 2(1 - x)$. Hence the desired ratio of the areas is

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABE)} = \frac{x^2}{1 - x} = 2.$$

21. **(B)** Let θ be the acute angle between the two lines. The area of shaded Region 1 in the diagram is

$$2\left(\frac{1}{2}\theta(1)^2\right) = \theta.$$



The area of shaded Region 2 is

$$2\left(\frac{1}{2}(\pi - \theta)(2^2 - 1^2)\right) = 3\pi - 3\theta.$$

The area of shaded Region 3 is

$$2\left(\frac{1}{2}\theta(3^2 - 2^2)\right) = 5\theta.$$

Hence the total area of the shaded regions is $3\pi + 3\theta$. The area bounded by the largest circle is 9π, so

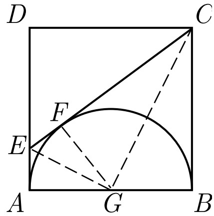
$$\frac{3\pi + 3\theta}{9\pi} = \frac{8}{8 + 13}.$$

Solving for θ gives $\theta = \pi/7$.

22. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2-x)^2 + 2^2 = (2+x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



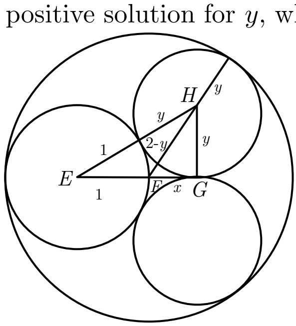
23. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1+y)^2 = (1+x)^2 + y^2 \quad \text{and} \quad (2-y)^2 = x^2 + y^2,$$

from which it follows that

$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 - \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (-2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.



24. **(D)** Note that

$$\begin{aligned} a_{2^1} &= a_2 = a_{2\cdot 1} = 1 \cdot a_1 = 2^0 \cdot 2^0 = 2^0, \\ a_{2^2} &= a_4 = a_{2\cdot 2} = 2 \cdot a_2 = 2^1 \cdot 2^0 = 2^1, \\ a_{2^3} &= a_8 = a_{2\cdot 4} = 4 \cdot a_4 = 2^2 \cdot 2^1 = 2^{1+2}, \\ a_{2^4} &= a_{16} = a_{2\cdot 8} = 8 \cdot a_8 = 2^3 \cdot 2^{1+2} = 2^{1+2+3}, \end{aligned}$$

and, in general, $a_{2^n} = 2^{1+2+\cdots+(n-1)}$. Because

$$1 + 2 + 3 + \cdots + (n-1) = \frac{1}{2}n(n-1),$$

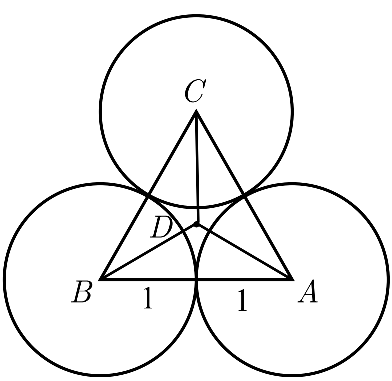
we have $a_{2^{100}} = 2^{(100)(99)/2} = 2^{4950}$.

25. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



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