The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions
Presented by The Akamai Foundation

53rd Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 12, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. **(A)** Factor to get $(2x+3)(2x-10)=0$, so the two roots are $-3/2$ and 5, which sum to $7/2$.

2. **(A)** Let x be the number she was given. Her calculations produce

$$\frac{x-9}{3}=43,$$

so

$$x-9=129 \quad \text{and} \quad x=138.$$

The correct answer is

$$\frac{138-3}{9}=\frac{135}{9}=15.$$

3. **(B)** No matter how the exponentiations are performed, 2^{2^2} always gives 16. Depending on which exponentiation is done last, we have

$$\left(2^{2^2}\right)^2=256, \quad 2^{\left(2^{2^2}\right)}=65,536, \quad \text{or} \quad \left(2^2\right)^{\left(2^2\right)}=256,$$

so there is one other possible value.

4. **(B)** The appropriate angle x satisfies

$$90-x=\frac{1}{4}(180-x), \quad \text{so} \quad 360-4x=180-x.$$

Solving for x gives $3x=180$, so $x=60$.

5. **(C)** The large circle has radius 3, so its area is $\pi \cdot 3^2=9\pi$. The seven small circles have a total area of $7\left(\pi \cdot 1^2\right)=7\pi$. So the shaded region has area $9\pi-7\pi=2\pi$.

6. **(E)** When $n=1$, the inequality becomes $m \le 1+m$, which is satisfied by all integers m. Thus, there are infinitely many of the desired values of m.

7. **(A)** Let $C_A=2\pi R_A$ be the circumference of circle A, let $C_B=2\pi R_B$ be the circumference of circle B, and let L the common length of the two arcs. Then

$$\frac{45}{360}C_A=L=\frac{30}{360}C_B.$$

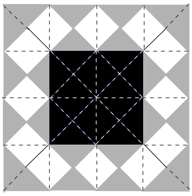
Therefore

$$\frac{C_A}{C_B}=\frac{2}{3} \quad \text{so} \quad \frac{2}{3}=\frac{2\pi R_A}{2\pi R_B}=\frac{R_A}{R_B}.$$

Thus, the ratio of the areas is

$$\frac{\text{Area of Circle }(A)}{\text{Area of Circle }(B)}=\frac{\pi R_A^2}{\pi R_B^2}=\left(\frac{R_A}{R_B}\right)^2=\frac{4}{9}.$$

8. **(A)** Draw additional lines to cover the entire figure with congruent triangles. There are 24 triangles in the blue region, 24 in the white region, and 16 in the red region. Thus, $B = W$.



9. **(B)** First note that the amount of memory needed to store the 30 files is

$$3(0.8) + 12(0.7) + 15(0.4) = 16.8 \text{ mb},$$

so the number of disks is at least

$$\frac{16.8}{1.44} = 11 + \frac{2}{3}.$$

However, a disk that contains a 0.8-mb file can, in addition, hold only one 0.4-mb file, so on each of these disks at least 0.24 mb must remain unused. Hence, there is at least $3(0.24) = 0.72$ mb of unused memory, which is equivalent to half a disk. Since

$$\left(11 + \frac{2}{3}\right) + \frac{1}{2} > 12,$$

at least 13 disks are needed.
To see that 13 disks suffice, note that:
Six disks could be used to store the 12 files containing 0.7 mb;
Three disks could be used to store the three 0.8-mb files together with three of the 0.4-mb files;
Four disks could be used to store the remaining twelve 0.4-mb files.

10. **(D)** After the first transfer, the first cup contains two ounces of coffee, and the second cup contains two ounces of coffee and four ounces of cream. After the second transfer, the first cup contains $2 + (1/2)(2) = 3$ ounces of coffee and $(1/2)(4) = 2$ ounces of cream. Therefore, the fraction of the liquid in the first cup that is cream is $2/(2 + 3) = 2/5$.

11. **(B)** Let t be the number of hours Mr. Bird must travel to arrive on time. Since three minutes is the same as 0.05 hours, $40(t + 0.05) = 60(t - 0.05)$. Thus,

$$40t + 2 = 60t - 3, \quad \text{so } t = 0.25.$$

The distance from his home to work is $40(0.25 + 0.05) = 12$ miles. Therefore, his average speed should be $12/0.25 = 48$ miles per hour.

OR

Let d be the distance from Mr. Bird's house to work, and let s be the desired average speed. Then the desired driving time is d/s. Since $d/60$ is three minutes too short and $d/40$ is three minutes too long, the desired time must be the average, so

$$\frac{d}{s} = \frac{1}{2}\left(\frac{d}{60} + \frac{d}{40}\right).$$

This implies that $s = 48$.

12. **(B)** Let p and q be two primes that are roots of $x^2 - 63x + k = 0$. Then

$$x^2 - 63x + k = (x - p)(x - q) = x^2 - (p + q)x + p \cdot q,$$

so $p + q = 63$ and $p \cdot q = k$. Since 63 is odd, one of the primes must be 2 and the other 61. Thus, there is exactly one possible value for k, namely $k = p \cdot q = 2 \cdot 61 = 122$.

13. **(C)** A number x differs by one from its reciprocal if and only if $x - 1 = 1/x$ or $x + 1 = 1/x$. These equations are equivalent to $x^2 - x - 1 = 0$ and $x^2 + x - 1 = 0$. Solving these by the quadratic formula yields the positive solutions

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{-1 + \sqrt{5}}{2},$$

which are reciprocals of each other. The sum of the two numbers is $\sqrt{5}$.

14. **(D)** We have

$$N = \log_{2002} 11^2 + \log_{2002} 13^2 + \log_{2002} 14^2 = \log_{2002} 11^2 \cdot 13^2 \cdot 14^2 = \log_{2002} (11 \cdot 13 \cdot 1$$

Simplifying gives

$$N = \log_{2002} (11 \cdot 13 \cdot 14)^2 = \log_{2002} 2002^2 = 2.$$

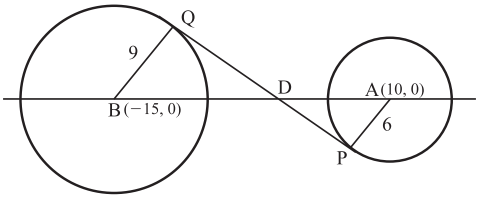
15. **(D)** The values 6, 6, 6, 8, 8, 8, 8, 14 satisfy the requirements of the problem, so the answer is at least 14. If the largest number were 15, the collection would have the ordered form 7, __, __, 8, 8, __, __, 15. But $7 + 8 + 8 + 15 = 38$, and a mean of 8 implies that the sum of all values is 64. In this case, the four missing values would sum to $64 - 38 = 26$, and their average value would be 6.5. This implies that at least one would be less than 7, which is a contradiction. Therefore, the largest integer that can be in the set is 14.

16. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is 1/10. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

$$\left(\frac{1}{10}\right)\left(1 + \frac{9}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} + 0 + 0 + 0\right) = \frac{2}{5}.$$

17. **(B)** First, observe that 4, 6, and 8 cannot be the units digit of any two-digit prime, so they must contribute at least $40 + 60 + 80 = 180$ to the sum. The remaining digits must contribute at least $1 + 2 + 3 + 5 + 7 + 9 = 27$ to the sum. Thus, the sum must be at least 207, and we can achieve this minimum only if we can construct a set of three one-digit primes and three two-digit primes. Using the facts that nine is not prime and neither two nor five can be the units digit of any two-digit prime, we can construct the sets $\{2, 3, 5, 41, 67, 89\}$, $\{2, 3, 5, 47, 61, 89\}$, or $\{2, 5, 7, 43, 61, 89\}$, each of which yields a sum of 207.

18. **(C)** The centers are at $A = (10, 0)$ and $B = (-15, 0)$, and the radii are 6 and 9, respectively. Since the internal tangent is shorter than the external tangent, $\overline{PQ}$ intersects $\overline{AB}$ at a point D that divides $\overline{AB}$ into parts proportional to the radii. The right triangles $\triangle APD$ and $\triangle BQD$ are similar with ratio of similarity $2 : 3$. Therefore, $D = (0, 0)$, $PD = 8$, and $QD = 12$. Thus $PQ = 20$.



19. **(D)** The equation $f(f(x)) = 6$ implies that $f(x) = -2$ or $f(x) = 1$. The horizontal line $y = -2$ intersects the graph of f twice, so $f(x) = -2$ has two solutions. Similarly, $f(x) = 1$ has 4 solutions, so there are 6 solutions of $f(f(x)) = 6$.

20. **(C)** Since $0.\overline{ab} = \frac{ab}{99}$, the denominator must be a factor of $99 = 3^2 \cdot 11$. The factors of 99 are 1, 3, 9, 11, 33, and 99. Since a and b are not both nine, the denominator cannot be 1. By choosing a and b appropriately, we can make fractions with each of the other denominators.

21. **(B)** Writing out more terms of the sequence yields

$$4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3,\ 4,\ 7,\ 1\ldots.$$

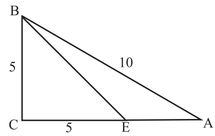
The sequence repeats itself, starting with the 13th term. Since $S_{12} = 60$, $S_{12k} = 60k$ for all positive integers k. The largest k for which $S_{12k} \le 10,000$ is

$$k = \left\lfloor \frac{10,000}{60} \right\rfloor = 166,$$

and $S_{12\cdot 166} = 60 \cdot 166 = 9960$. To have $S_n > 10,000$, we need to add enough additional terms for their sum to exceed 40. This can be done by adding the next 7 terms of the sequence, since their sum is 42. Thus, the smallest value of n is $12 \cdot 166 + 7 = 1999$.

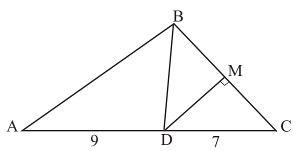
22. **(C)** Since AB is 10, we have $BC = 5$ and $AC = 5\sqrt{3}$. Choose E on $\overline{AC}$ so that $CE = 5$. Then $BE = 5\sqrt{2}$. For BD to be greater than $5\sqrt{2}$, P has to be inside $\triangle ABE$. The probability that P is inside $\triangle ABE$ is

$$\frac{\text{Area of } \triangle ABE}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}EA \cdot BC}{\frac{1}{2}CA \cdot BC} = \frac{EA}{AC} = \frac{5\sqrt{3}-5}{5\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}.$$



23. **(D)** By the angle-bisector theorem, $\frac{AB}{BC} = \frac{9}{7}$. Let $AB = 9x$ and $BC = 7x$, let $m\angle ABD = m\angle CBD = \theta$, and let M be the midpoint of $\overline{BC}$. Since M is on the perpendicular bisector of $\overline{BC}$, we have $BD = DC = 7$. Then

$$\cos\theta = \frac{\frac{7x}{2}}{7} = \frac{x}{2}.$$



Applying the Law of Cosines to $\triangle ABD$ yields

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7)\left(\frac{x}{2}\right),$$

from which $x = 4/3$ and $AB = 12$. Apply Heron's formula to obtain the area of triangle ABD as $\sqrt{14 \cdot 2 \cdot 5 \cdot 7} = 14\sqrt{5}$.

24. **(E)** Let $z = a + bi$, $\overline{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \overline{z}$. Note that

$$|z|^{2002} = \left|z^{2002}\right| = |\overline{z}| = |z|,$$

from which it follows that

$$|z|\left(|z|^{2001} - 1\right) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \overline{z}$, which is equivalent to $z^{2003} = \overline{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

25. **(B)** The sum of the coefficients of P and the sum of the coefficients of Q will be equal, so $P(1) = Q(1)$. The only answer choice with an intersection at $x = 1$ is (B). (The polynomials in graph B are $P(x) = 2x^4 - 3x^2 - 3x - 4$ and $Q(x) = -2x^4 - 2x^2 - 2x - 2$.)

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