

The Remainder Theorem says:

1. When a polynomial, $f(x)$, is divided by $x - a$, the remainder is equal to $f(a)$.
2. When a polynomial, $f(x)$, is divided by $ax - b$, the remainder is equal to $f\left(\frac{b}{a}\right)$.

Example 1:

Determine the remainder when $x^3 + 7x^2 + 2x - 5$ is divided by $x + 7$.

Define a function: let $f(x) = x^3 + 7x^2 + 2x - 5$

$$f(x) = (x + 7)Q(x) + R$$

$$R = f(-7)$$

$$R = (-7)^3 + 7(-7)^2 + 2(-7) - 5$$

$$\Delta R = -19$$

Example 2:

For what value of b will the polynomial $f(x) = -2x^3 + bx^2 - 5x + 2$ have the same remainder when divided by $x - 2$ and $x + 1$?

$$R_1 = f(2), \quad R_2 = f(-1)$$

$f(2) = f(-1)$ since remainders are equal

$$f(2) = -2(2)^3 + b(2)^2 - 5(2) + 2 \rightarrow f(2) = -16 + 4b - 8 = -24 + b$$

$$f(-1) = -2(-1)^3 + b + 5 + 2 \rightarrow f(-1) = b + 9$$

\uparrow
 $-5(-1)$

$$4b - 24 = b + 9$$

$$4b - b = 24 + 9, \quad 3b = 33, \quad b = 11$$

Example 3:

When $x^3 + 3x^2 - kx + 10$ is divided by $x - 5$, the remainder is 15. Find the value of k .

$$\text{let } f(x) = x^3 + 3x^2 - kx + 10$$

$$\begin{cases} R = f(5) = (5)^3 + 3(5)^2 - k(5) + 10 \\ R = 15 \end{cases}$$

$$210 - 5k = 15$$

$$5k = 210 - 15$$

$$5k = 195$$

$$k = 39 \checkmark$$

Example 4:

When $2x^3 - mx^2 + nx - 2$ is divided by $x+1$, the remainder is -12 . The binomial $x-2$ is a factor.

Determine the values of m and n .

let $f(x) = 2x^3 - mx^2 + nx - 2$

① If divided by $(x+1)$,

$$R = -12$$

$$f(-1) = -12$$

$$f(-1) = 2(-1)^3 - m(-1)^2 + n(-1) - 2 = -12$$

$$-m - n - 4 = -12$$

$$-m - n = -12 + 4$$

$$\boxed{m + n = 8}$$

② If divided by $x-2$, $R=0$ ("is a factor")

$$f(2) = 0$$

$$2(2)^3 - m(2)^2 + n(2) - 2 = 0$$

$$16 - 4m + 2n - 2 = 0$$

$$2n - 4m = -14$$

$$\boxed{n - 2m = -7}$$

We now solve a linear system:

$$\begin{cases} n + m = 8 & \longrightarrow 2m + 2n = 16 & \leftarrow \text{add equations} \\ n - 2m = -7 & \longrightarrow \underline{n - 2m = -7} & \leftarrow \text{to eliminate } m. \end{cases}$$
$$3n = 9$$
$$n = 3$$

$$\text{then } 3 + m = 8$$

$$m = 8 - 3$$

$$m = 5$$

Answer: $m=5$, $n=3$

Answers:

1. $R = -19$

2. $b = 11$

3. $k = 39$

4. $m = 5$ and $n = 3$