p. 558 #15

Determine the equation of the line that passes through the point $A(1,0,2)$ and intersects the line $\vec{r}=(-2,3,4)+s(1,1,2)$, $s\in\mathbb{R}$, at a right angle.

Solution: let $\vec{d}=(a,b,c)$ represent the direction vector for this line.

Then $\vec{r}=(1,0,2)+t(a,b,c)$, $t\in\mathbb{R}$

$$(a,b,c)\cdot(1,1,2)=a+b+2c=0$$

$$b=-a-2c.$$

POI: Since $(1,0,2)$ is not on the other line we may choose a, b, and c such the intersection will occur at $t=1$,

the POI is then

$$(1,0,2)+(a,b,c)=(1+a,b,2+c)$$

and $b=-a-2c$

$$(1+a,b,2+c)=(1+a,-a-2c,2+c)$$

For some s-value,

$$x=-2+s=1+a \quad ①$$

$$y=3+s=-a-2c \quad ②$$

$$z=4+2s=2+c \quad ③$$

Now ② − ①: $5=-2a-2c-1$

$$6=-2a-2c$$

$$a+c=-3$$

Then ③ − 2×①: $8=c-2a$ and $\begin{cases} a+c=-3 \\ c-2a=8 \end{cases} \rightarrow c=\frac{2}{3}$, $a=-\frac{11}{3}$

$$b=-a-2c=-\left(-\frac{11}{3}\right)+2\left(\frac{2}{3}\right)=\frac{7}{3}$$

$(a,b,c)=\left(-\frac{11}{3},\frac{7}{3},\frac{2}{3}\right)$ A direction vector for the line is then $(-11,7,2)$

Then $(1,0,2)+\underbrace{(a,b,c)}_{\text{literally as solved for}}=\left(\frac{3}{3},\frac{0}{3},\frac{6}{3}\right)+\left(-\frac{11}{3},\frac{7}{3},\frac{2}{3}\right)=\left(-\frac{8}{3},\frac{7}{8},\frac{8}{3}\right)$

is a point on the line. We check that:

$x=-2+s=-\frac{8}{3}$, $s=-\frac{2}{3}$.

$\vec{r}=(-2,3,4)+\left(-\frac{2}{3}\right)(1,1,2)=\left(-\frac{8}{7},\frac{7}{3},\frac{8}{3}\right)\Rightarrow\vec{q}=(1,0,2)+t(-11,7,2)$

$t\in\mathbb{R}$ is the line intersecting the given line at a right angle.