Divisor Counting

Ex2: $192 = 2^6 \cdot 3^1$ ✓

① PND:

192	2
96	2
48	2
24	2
12	2
6	2
3	3
1	

② Any divisor of 192 is of the form

$2^a \cdot 3^b$

a: 0, 1, 2, 3, 4, 5, 6 ← 7 possibilities.

↑ account for zero.

b: 0, 1 ← 2 possibilities.

Principle of multiplying Possibilities.

$\underbrace{t(192)}_{\text{function notation}} = \text{"t of 192"} = (2)(7) = 14$

$t(192) = (6+1)(1+1) = 14$

$N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots \cdot p_n^{e_n}$

$t(N) = (e_1+1)(e_2+1)(e_3+1) \cdot \ldots \cdot (e_n+1)$

Problem 1: How many of the positive divisors of 168 are even?

168	2
84	2
42	2
21	3
7	7
1	

method 1: Direct Method.

$168 = 2^3 \cdot 3 \cdot 7$

Any divisor of 168 is of the form $2^a \cdot 3^b \cdot 7^c$

Any even divisor of 168 is of the form $2^a \cdot 3^b \cdot 7^c$ where

a: 1, 2, 3 ← 3 possib

b: 0, 1 ← 2 possib

c: 0, 1 ← 2 possib.

$e \in \mathbb{W} = \{0, \underbrace{1, 2, 3, \ldots}_{\mathbb{N}}$

The number of even divisors of 168 = (3)(2)(2)

= 12 ✓

Method 2: Indirect – Complementary Counting.

We calculate the number of odd divisors first.

Any odd divisor of 168 is of the form

$$2^0 \cdot 3^b \cdot 7^c = 3^b \cdot 7^c$$

b: 0, 1 $\leftarrow$ 2 possib

c: 0, 1 $\leftarrow$ 2 possib.

(Total) number of odd divisors of 168 = (2)(2) = 4 ✓

$$t(168) = (3+1)(1+1)(1+1) = 16$$

number of even divisors is

$$= 16 - 4$$

$$= 12$$ ✓

Prob 2:

Prove that any perfect square has an odd number of divisors.

$$N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots \cdot p_k^{e_k}; \quad e_i \in \mathbb{W} = \{0, 1, 2, 3, \ldots\}, \; i \in \mathbb{N}$$

$$N^2 = p_1^{2e_1} \cdot p_2^{2e_2} \cdot p_3^{2e_3} \cdot \ldots \cdot p_k^{2e_k}$$

$$t(N^2) = \underbrace{(2e_1+1)}_{\text{odd \#}} \underbrace{(2e_2+1)}_{\text{another odd \#}} \underbrace{(2e_3+1)}_{\text{yet another odd \#}} \cdot \ldots \cdot (2e_k+1)$$

$\therefore$ $t(N^2)$ is an odd integer as a product of odd integers (having no factor of 2 in them)

Remark: $a \in \mathbb{Z}, a \geq 0, b \in \mathbb{Z}, b \geq 0$

$$(2a+1)(2b+1)$$
$$= 4ab + 2a + 2b + 1$$
$$= 2(2ab + a + b) + 1$$
$$= 2x + 1, \quad x = 2ab + a + b, \; x \in \mathbb{Z}, \; x \geq 0.$$