

**Prime and Composite Natural Numbers.**

A factor of a number is a positive integer that goes evenly (without remainder) into the given number.

Example: number 20 has six divisors/factors, which are: 1, 2, 4, 5, 10, 20.  $t(20)=6$  ✓

A natural number, other than 1, is called a prime number if it has only two divisors: 1 and the number itself. There exists the smallest prime number, which is: 2. This is the only prime number that is even.

Any even number greater than 2 is, therefore a composite number. There is no largest prime number as there are infinitely many prime numbers. A natural number that has more than two divisors is called a composite number.   
 ↳ proof?

Any composite number  $N$  is divisible by some number that is less than or equal to the square root of  $N$ .

Why? Divisors come in pairs. If  $N$  is a perfect square, then  $\sqrt{N}$  is a divisor and other divisors are less than or greater so that each divisor less than  $\sqrt{N}$  is paired with a divisor greater than  $\sqrt{N}$ . Therefore, it enough to check numbers (primes) less than  $\sqrt{N}$ . ✓

If  $N$  is a non-square, then  $\sqrt{N}$  is not a divisor of  $N$ . The divisors of  $N$  are either less than  $\sqrt{N}$  or greater than  $\sqrt{N}$  and each divisor less than  $\sqrt{N}$  is paired up with a divisor greater than  $\sqrt{N}$ .

That gives us a way of checking if a number is a prime.

Example: Check if 137 is a prime.

$\sqrt{137} = 11.7 \dots$  ?

Primes: ~~2, 3, 5, 7, 11~~

∴ 137 is a prime. ✓

**Problem:**

Prime numbers have only two divisors: number 1 and the number itself.

What numbers have exactly 3 different divisors? Perfect Squares of Primes.

[Hint: To every divisor  $m$  of the dividend  $M$  there corresponds another divisor  $\frac{M}{m}$ , in other words divisors come

in pairs].

What numbers have exactly 4 different divisors?

Type 1: Perfect Cube of a Prime  
 $p^3: 1, p, p^2, p^3$

Type 2: Product of Two Primes  
 $pq: 1, p, q, pq$

**The Fundamental Theorem of Arithmetic:**

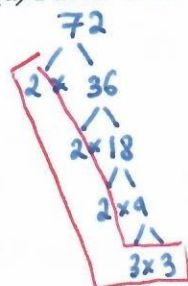
Any natural number  $N$  greater than 1 can be expressed as a product of primes. ✓  
 This representation is unique, except for the order of factors. Then we have a record of the type

$N = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$

The number of prime divisors of  $N$  according to this record is  $k$ . For example,  $2^3 3^2 5^9$  has 3 prime divisors.

Example: Factorization of a Natural Number into Prime Factors. Determine the prime number decomposition of 72.

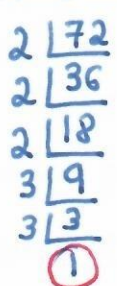
(a) Factor Tree



(b) Column Method



(c) Step Method



PND of 72:  
 $72 = 2^3 \cdot 3^2$

**PND of Perfect Squares.**

$$N = p_1^{e_1} \cdot p_2^{e_2}; \quad N^2 = (p_1^{e_1} \cdot p_2^{e_2})^2 = (p_1^{e_1})^2 \cdot (p_2^{e_2})^2 = p_1^{2e_1} \cdot p_2^{2e_2}$$

Perfect squares have even exponents on primes in their decomposition. That means the smallest exponent on a prime is 2. That means, for example, that if a square is divisible by 2, it is divisible by 4.

**PND of Perfect Cubes.**  $N = p_1^{e_1} \cdot p_2^{e_2}; \quad N^3 = (p_1^{e_1} \cdot p_2^{e_2})^3 = p_1^{3e_1} \cdot p_2^{3e_2}$

Perfect cubes have exponents that are multiples of three on primes in their decomposition.

**Practice:**

- Determine the prime number decomposition of the following numbers:  
 (a) 360                      (b) 911250                      (c) 792                      (d) 132                      (e) 3780
- Why are the following numbers not perfect squares 432, 21125, 361 and 14175.
- Determine two two-digit numbers, AX and YX such that  $AX \cdot YX = 2001$
- The number 13 is a prime. If you reverse the digits you also obtain a prime number, 31. What is the largest of the pair of primes that satisfies this condition and has a sum of 110?
- Find the largest possible product of three prime numbers that add up to 30.
- How many two-digit prime numbers can be written as a sum of two primes?
- If a and b are positive integers, neither of which is divisible by 10 and  $ab = 10000$ , what is the sum of a and b? [Answer: 641]
- The product of integers 240 and k is a perfect cube. What is the smallest positive value of k?
- If k is an integer and  $k > 100$ , what is the smallest possible integer value of the cube root of  $k^2$ ?
- If p is a prime, different from 2, show that  $p^2 + 13$  is composite.
- If positive integers p, q are both prime, what is the least possible value of  $pq$ ?
- What is the smallest prime divisor of  $5^{23} + 7^{17}$ ?

**Hints/Answers:**

- (a)  $360 = 2^3 \cdot 3^2 \cdot 5$                       (b)  $911250 = 2 \cdot 3^6 \cdot 5^4$                       (c)  $792 = 2^3 \cdot 3^2 \cdot 11$   
 (d)  $132 = 2^2 \cdot 3 \cdot 11$                       (e)  $3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7$
- Hint: consider the PND of each number. What would need to happen for the number to be a perfect square?
- The numbers are 69 and 29.                      4. 73                      5. 374                      6. 6 primes.
- 641
- Answer: 900.
- Answer: 25. The cube root of  $k^2$  has to be an integer. That means  $k^2$  is a .... and, therefore, k itself is that.
- Answer: 2