MPM2DG **Prime and Composite Natural Numbers.**

A ~~divisor~~ factor of a number is a positive integers natural number that goes evenly (without remainder) into the given number.

Example: number 20 has six divisors/factors, which are: 1, 2, 4, 5, 10, 20. $t(20)=6$ ✓

A natural number, other than 1, is called a prime number if it has only two divisors: 1 and the number itself.
There exists the smallest prime number, which is: 2. This is the only prime number that is even.

Any even number greater than 2 is, therefore a composite number. There is no largest prime number as there are infinitely many prime numbers. A natural number that has more than two divisors is called a composite number. ↳ proof?

Any composite number N is divisible by some number that is less than or equal to the square root of N.

Why? Divisors come in pairs. If N is a perfect square, then $\sqrt{N}$ is a divisor and other divisors are less than or greater so that each divisor less than $\sqrt{N}$ is paired with a divisor greater than $\sqrt{N}$. Therefore, it enough to check numbers (primes) less then $\sqrt{N}$. ✓

If N is a non-square, then $\sqrt{N}$ is not a divisor of N. The divisors of N are either less than $\sqrt{N}$ or greater than $\sqrt{N}$ and each divisor less than $\sqrt{N}$ is paired up with a divisor greater than $\sqrt{N}$.

That gives us a way of checking if a number is a prime.

Example: Check if 137 is a prime. $\sqrt{137} = 11.?$ Primes: ~~2, 3, 5, 7, 11~~
∴ 137 is a prime. ✓

Problem:
Prime numbers have only two divisors: number 1 and the number itself.
What numbers have exactly 3 different divisors? Perfect Squares of Primes.

[Hint: To every divisor m of the dividend M there corresponds another divisor $\frac{M}{m}$, in other words divisors come in pairs].

What numbers have exactly 4 different divisors?

Type 1: Perfect Cube of a Prime
p^3: $1, p, p^2, p^3$.

Type 2: Product of Two Primes
pq: $1, p, q, pq$.

The Fundamental Theorem of Arithmetic:
Any natural number N greater than 1 can be expressed as a product of primes. ✓
This representation is unique, except for the order of factors. Then we have a record of the type

$$N = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

The number of prime divisors of N according to this record is k. For example, $2^3 3^2 5^9$ has 3 prime divisors.

Example: Factorization of a Natural Number into Prime Factors.
Determine the prime number decomposition of 72.

(a) Factor Tree

72
2 × 36
2 × 18
2 × 9
3 × 3

(b) Column Method

72	2
36	2
18	2
9	3
3	3
1	

(c) Step Method

2 | 72
2 | 36
2 | 18
3 | 9
3 | 3
1

PND of 72:
$72 = 2^3 \cdot 3^2$

PND of Perfect Squares. $N = p_1^{e_1} \cdot p_2^{e_2}$; $N^2 = (p_1^{e_1} \cdot p_2^{e_2})^2 = (p_1^{e_1})^2 \cdot (p_2^{e_2})^2 = p_1^{2e_1} \cdot p_2^{2e_2}$

Perfect squares have even exponents on primes in their decomposition. That means the smallest exponent on a prime is 2. That means, for example, that if a square is divisible by 2, it is divisible by 4.

PND of Perfect Cubes. $N = p_1^{e_1} \cdot p_2^{e_2}$; $N^3 = (p_1^{e_1} \cdot p_2^{e_2})^3 = p_1^{3e_1} \cdot p_2^{3e_2}$

Perfect cubes have exponents that are multiples of three. on primes in their decomposition.

Practice:

1. Determine the prime number decomposition of the following numbers:

   (a) 360 (b) 911250 (c) 792 (d) 132 (e) 3780

2. Why are the following numbers not perfect squares 432, 21125, 361 and 14175.

3. Determine two two-digit numbers, AX and YX such that $AX \cdot YX = 2001$

4. The number 13 is a prime. If you reverse the digits you also obtain a prime number, 31. What is the largest of the pair of primes that satisfies this condition and has a sum of 110?

5. Find the largest possible product of three prime numbers that add up to 30.

6. How many two-digit prime numbers can be written as a sum of two primes?

7. If a and b are positive integers, neither of which is divisible by 10 and $ab = 10000$, what is the sum of a and b? [Answer: 641]

8. The product of integers 240 and k is a perfect cube. What is the smallest positive value of k?

9. If k is an integer and $k > 100$, what is the smallest possible integer value of the cube root of k^2?

10. If p is a prime, different from 2, show that $p^2 + 13$ is composite.

11. If positive integers p, q are both prime, what is the least possible value of pq?

12. What is the smallest prime divisor of $5^{23} + 7^{17}$?

Hints/Answers:

1. (a) $360 = 2^3 \cdot 3^2 \cdot 5$ (b) $911250 = 2 \cdot 3^6 \cdot 5^4$ (c) $792 = 2^3 \cdot 3^2 \cdot 11$

(d) $132 = 2^2 \cdot 3 \cdot 11$ (e) $3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7$

2. Hint: consider the PND of each number. What would need to happen for the number to be a perfect square?

3. The numbers are 69 and 29. **4.** 73 **5.** 374 **6.** 6 primes.

7. 641

8. Answer: 900.

9. Answer: 25. The cube root of k^2 has to be an integer. That means k^2 is a .... and, therefore, k itself is that.

12. Answer: 2

Compiled by M.Timmerman. Modified March 2018