MPM2DG **Divisor Counting Problems.**

Problem 1: How many of the positive divisors of 168 are even?

Direct solution: First we learn what 168 is made of - prime number decomposition.

168
2 × 84
2 × 2 × 42
2 × 2 × 2 × 21
2 × 2 × 2 × 3 × 7

Then $168 = 2^3 \cdot 3 \cdot 7$. Any divisor of 168 has the form: $d = 2^a \cdot 3^b \cdot 7^c$

where $a = 0, 1, 2, \text{or } 3$
$b = 0 \text{ or } 1$
$c = 0 \text{ or } 1$.

If divisors are to be even, then a must be at least 1:

$3 \times 2 \times 2 = 12$

possibilities for a
possibilities for c
possibilities for b

$\boxed{\text{Answer: 12 divisors of 168 are even}}$

Complimentary Counting solution

Total things − things we don't need = things we do need.

$t(168) = (3+1)(1+1)(1+1) = 16$. If a divisor is not even, it's odd.

Number of odd divisors:

$d = 2^0 \cdot 3^b \cdot 7^c$ or $d = 3^b \cdot 7^c$ where $b = 0 \text{ or } 1$, $c = 0 \text{ or } 1$

$2 \times 2 = 4 \rightarrow$ There are four odd divisors

number of even divisors $= 16 - 4 = 12$ ← same answer as before :)

Problem 2: Show that any positive perfect square has an odd number of positive divisors.

let n be a natural number.

$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_m^{e_m}$ ← prime number decomposition.

We now construct a perfect square by squaring n:

$$N = n^2 = (p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_m^{e_m})^2 = (p_1^{e_1})^2 \cdot (p_2^{e_2})^2 \cdot \ldots \cdot (p_m^{e_m})^2 = p_1^{2e_1} \cdot p_2^{2e_2} \cdot \ldots \cdot p_m^{2e_m}$$

Using the formula for total number of positive integer divisors, we get

$$t(N) = t(n^2) = \underbrace{(2e_1+1)}_{\text{odd}}\underbrace{(2e_2+1)}_{\text{odd}} \cdot \ldots \cdot \underbrace{(2e_m+1)}_{\text{odd}};$$ $e_1, e_2, \ldots, e_m$ are integers

$t(N)$ is odd as it is a product of odd numbers.

Remark: The prime factorization of a natural number that is not a perfect square must include some prime number raised to an odd exponent so that non-squares have an even number of positive divisors.