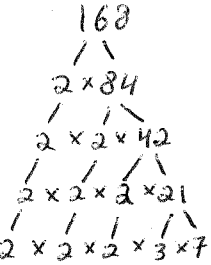


Problem 1: How many of the positive divisors of 168 are even?

Direct solution: First we learn what 168 is made of - prime number decomposition.



Then  $168 = 2^3 \cdot 3 \cdot 7$ . Any divisor of 168 has the form:  $d = 2^a \cdot 3^b \cdot 7^c$

where  $a = 0, 1, 2, \text{ or } 3$   
 $b = 0 \text{ or } 1$   
 $c = 0 \text{ or } 1$

If divisors are to be even, then  $a$  must be at least 1.

$3 \times 2 \times 2 = 12$   
 ↑ possibilities for  $a$   
 ↑ possibilities for  $b$   
 ↑ possibilities for  $c$

Answer: 12 divisors of 168 are even

Complimentary Counting solution

Total things - things we don't need = things we do need.

$t(168) = (3+1)(1+1)(1+1) = 16$ . If a divisor is not even, it's odd.

Number of odd divisors:

$d = 2^0 \cdot 3^b \cdot 7^c$  or  $d = 3^b \cdot 7^c$  where  $b = 0 \text{ or } 1$ ,  $c = 0 \text{ or } 1$

$2 \times 2 = 4 \rightarrow$  There are four odd divisors

number of even divisors =  $16 - 4 = 12$  ← same answer as before :)

Problem 2: Show that any positive perfect square has an odd number of positive divisors.

let  $n$  be a natural number.

$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_m^{e_m}$  ← prime number decomposition.

We now construct a perfect square by squaring  $n$ :

$$N = n^2 = (p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_m^{e_m})^2 = (p_1^{e_1})^2 \cdot (p_2^{e_2})^2 \cdot \dots \cdot (p_m^{e_m})^2 = p_1^{2e_1} \cdot p_2^{2e_2} \cdot \dots \cdot p_m^{2e_m}$$

Using the formula for total number of positive integer divisors, we get

$$t(N) = t(n^2) = \underbrace{(2e_1 + 1)}_{\text{odd}} \cdot \underbrace{(2e_2 + 1)}_{\text{odd}} \cdot \dots \cdot \underbrace{(2e_m + 1)}_{\text{odd}}; \quad e_1, e_2, \dots, e_m \text{ are integers}$$

$t(N)$  is odd as it is a product of odd numbers.

Remark: The prime factorization of a natural number that is not a perfect square must include some prime number raised to an odd exponent so that non-squares have an even number of positive divisors.