The Binomial Theorem III – ANSWERS

Ex. Expand $(x^2+2)\left(x-\frac{1}{x}\right)^5$ and collect like terms.

You don't have to write using sigma notation. With practice, some of us are very comfortable with the notation. Does it help?

$$(x^2+2)\left(\sum_{r=0}^{5} {}_5C_r\left(x^{5-r}\right)(-1)^r\left(x^{-r}\right)\right)$$

$$=(x^2+2)\left(\sum_{r=0}^{5}(-1)^r {}_5C_r\left(x^{5-r-r}\right)\right)$$

Use exponent laws to collect the x^n term.

$$=x^2\left(\sum_{r=0}^{5}(-1)^r {}_5C_r\left(x^{5-2r}\right)\right)+2\left(\sum_{r=0}^{5}(-1)^r {}_5C_r\left(x^{5-2r}\right)\right)$$

$$=\sum_{r=0}^{5}(-1)^r {}_5C_r\left(x^{5-2r}\right)x^2+\sum_{r=0}^{5}(-1)^r {}_5C_r(2)\left(x^{5-2r}\right)$$

Shim-tip!
Wrap-around & line them up to collect like-terms!

$$=\sum_{r=0}^{5}(-1)^r {}_5C_r\left(x^{7-2r}\right)+\sum_{r=0}^{5}(-1)^r {}_5C_r(2)\left(x^{5-2r}\right)$$

$$=x^7-5x^5+10x^3-10x+5x^{-1}-x^{-3}$$
$$\quad +2x^5-10x^3+20x-20x^{-1}+10x^{-3}-2x^{-5}$$
$$=x^7-3x^5+10x-15x^{-1}+9x^{-3}-2x^{-5}$$

Ex. Find the first three terms in the expansion of $(x+1)^5(x-2)^6$

Here, we only need $r = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$
Sigma notation doesn't help.

$$(x+1)^5(x-2)^6$$
$$=\left(\sum_{r=0}^{5} {}_5C_r x^{5-r}\right)\left(\sum_{k=0}^{6} {}_6C_k x^{6-k}(-2)^k\right)$$
$$=\left(x^5+5x^4+10x^3+\ldots\right)\left(x^6-12x^5+60x^4+\ldots\right)$$
$$=x^{11}-12x^{10}+60x^9+\ldots+5x^{10}-60x^9+300x^8+\ldots+10x^9-120x^8+600x^7+\ldots$$
$$=x^{11}-7x^{10}+10x^9+180x^8+600x^7\ldots$$

or, we could work with the general terms…

$t_r = {}_5C_r x^{5-r}$ $\qquad$ $t_k = {}_6C_k(-2)^k x^{6-k}$

* it is crucial to use different variables r and k (so you can tell which comes from the 1st or 2nd binomial)

$$t_r\cdot t_k = {}_5C_r x^{5-r}\cdot {}_6C_k(-2)^k x^{6-k}$$
$$= {}_5C_r\cdot {}_6C_k(-2)^k x^{5-r}x^{6-k}$$
$$= \underbrace{{}_5C_r\cdot {}_6C_k(-2)^k}_{\text{coefficient}} x^{11-r-k}$$

$r=0, k=0$ $\qquad$ $r=0, k=1;$ $r=1, k=0$ $\qquad$ $r=0, k=2;$ $r=2, k=0;$ $r=1; k=1$

$$[\quad]x^{11}+[\quad]x^{10}+[\quad]x^9+\ldots$$

The Binomial Theorem III – ANSWERS

Date:

Ex. In the expansion of $(x^2-1)^8(x+2)^7$ find the term containing x^8.

For $(x^2-1)^8$,

$$t_{n+1} = {}_8C_n(x^2)^{8-n}(-1)^n$$
$$= (-1)^n\,{}_8C_n x^{16-2n}, \quad n = 0, 1, 2, \ldots, 8$$

For $(x+2)^7$,

$$t_{k+1} = {}_7C_k(x^{7-k})(2^k)$$
$$= 2^k\,{}_7C_k x^{7-k}, \quad k = 0, 1, 2, \ldots, 7$$

Therefore, every term in the expansion is of the form $(-1)^n(2^k)({}_8C_n)({}_7C_k)x^{23-2n-k}$, $n = 0, \ldots, 8$, $k = 0, \ldots, 7$

Thus, $23 - 2n - k = 8$

$2n + k = 15$

Solve by listing numbers:

n	k
4	7
5	5
6	3
7	1

Now we see why the domain is important!

Substitute all possible (n, k) into the general term.

Therefore, the x^8 term is:

$$(-1)^4(2^7)({}_8C_4)({}_7C_7)x^{23-2(4)-7} + (-1)^5(2^5)({}_8C_5)({}_7C_5)x^{23-2(5)-5} + (-1)^6(2^3)({}_8C_6)({}_7C_3)x^{23-2(6)-3} + (-1)^7(2^1)({}_8C_7)({}_7C_1)x^{23-2(7)-1}$$

$$= (-112 + 7840 - 37\,632 + 8960)x^8$$
$$= -20\,944x^8$$