Introduction to Sigma Notation (notes)

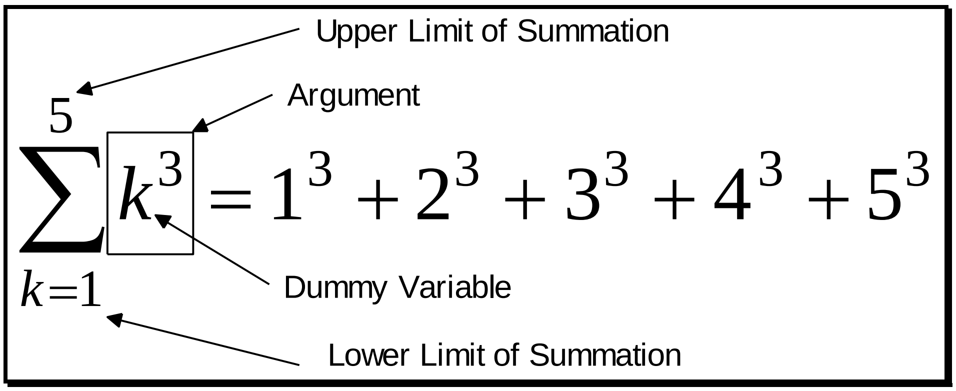
Date:

Sigma Notation is a quick way of writing ***a series*** which have a predictable pattern in their terms.

The three series given above can be represented by sigma notation as follows:

$$1+2+3+4+5=\sum_{i=1}^{5} i \qquad 5+10+15+\ldots+605+610=\sum_{i=1}^{122} 5i \qquad 1+2+3+\ldots+(n-1)+n=\sum_{i=1}^{n} i$$

The symbol Σ is the Greek capital "S". The parts of the sigma notation are shown in the example below:



The expression above is read as "*sigma for k from 1 to 5 of k^3*".

The ***argument*** is the expression which actually produces each of the terms of the series. Inside the argument is the ***dummy variable***. The dummy variable is a place-holder for the part of the expression which changes as each different term is produced. To produce each term, the dummy variable first takes on the value of the ***lower limit of summation*** – this creates the first term of the series. The dummy variable then increases by 1 to produce the next term, and keeps increasing in steps of 1 to produce all the other terms in the series. When the value of the dummy variable reaches the ***upper limit of summation***, it produces the last term of the series. The dummy variable ***always*** increases in steps of 1 from the lower to the upper limit of summation. Since the dummy variable is replaced by numbers as each term of the series is produced, it never appears in the expanded form of the series. The choice of letter for dummy variable is therefore of no importance to the final answer, though it is common practice to choose letters from the middle of the alphabet.

Carefully study each of the examples on the following pages and try to understand how the given sigma notation results in each of the series

<u>Examples:</u>

1. $\sum_{j=1}^{4} 3j = 3(1)+3(2)+3(3)+3(4)$
2. $\sum_{k=3}^{6} k = 3+4+5+6$
3. $\sum_{i=-1}^{2} (i+3) = (-1+3)+(0+3)+(1+3)+(2+3)$
4. $\sum_{i=0}^{2} 2^i = 2^0+2^1+2^2$
5. $\sum_{j=5}^{6} \left(x^j-2j+1\right) = \left(x^5-2(5)+1\right)+\left(x^6-2(6)+1\right)$
6. $\sum_{k=1}^{7} 4 = 4+4+4+4+4+4+4$

Notice in the preceding examples that the lower limit need not be 1, it may indeed be any integer as long as it is less than or equal to the upper limit of summation. Also note that, as in example #6, the dummy variable need not be present in the argument at all – in this case it merely acts as a counter, controlling the number of terms produced.

In the example below, each step is the instruction which would be given if you were required to produce an answer in that form. ***Take note of these instructions as they will be used in future assignments and tests.***

$\sum_{j=1}^{4} 3j = 3(1)+3(2)+3(3)+3(4)$ .... ***"write explicitly"***

$= 3 + 6 + 9 + 12$ .... ***"write explicitly, simplifying each term"***

$= 30$ .... ***"evaluate"***

Sigma notation many also have numbers outside to the left of the sigma. Brackets many be used to change the meaning of the expression as well. Study the examples below to better understand these issues.

i)	$\underbrace{\sum_{i=1}^{3}(2i+1)}_{\text{extent of sigma notation}} = \underbrace{(2(1)+1)+(2(2)+1)+(2(3)+1)}_{\text{produced by sigma notation}} = 3+5+7 = 15$
ii)	$2\underbrace{\sum_{i=1}^{3}(i+1)}_{\text{extent of sigma notation}} = 2(\underbrace{(1+1)+(2+1)+(3+1)}_{\text{produced by sigma notation}}) = 2(2+3+4) = 2(9) = 18$
iii)	$\underbrace{\sum_{i=1}^{3}2i}_{\text{extent of sigma notation}} + 1 = \underbrace{(2(1)+2(2)+2(3))}_{\text{produced by sigma notation}} + 1 = (2+4+6)+1 = 12+1 = 13$

It is especially important in the last of the above examples (iii) to understand that without the use of brackets, the argument of the sigma is only the $2i$ term – the "1" is not part of the sigma's argument and therefore is not repeated.

<u>Exercises:</u>

1. For each of the expressions below, show with ***one or more middle steps*** that each of them can be written explicitly (individual terms simplified) as: 4 + 7 + 10 + 13 + 16.

a) $\sum_{i=1}^{5}(3i+1)$ b) $\sum_{k=1}^{5}(3k+1)$ c) $\sum_{j=2}^{6}(3j-2)$

d) $\sum_{i=3}^{7}(3i-5)$ e) $\sum_{i=-2}^{2}(3i+10)$ f) $\sum_{i=1}^{3}(3i+1)+\sum_{i=4}^{5}(3i+1)$

g) $\sum_{i=1}^{4}(3i+1)+\sum_{i=5}^{5}(3i+1)$ h) $\sum_{i=1}^{3}(3i+1)+\sum_{i=2}^{3}(3i+7)$

i) $\sum_{i=3}^{7}(3i+1)-\sum_{i=6}^{7}(3i+1)+\sum_{i=1}^{2}(3i+1)$ j) $\sum_{i=1}^{3}(6i-2)+\sum_{i=-5}^{-4}(6i+37)$

2. Show that the following expressions, while not producing the same pattern: 4 + 7 + 10 + 13 + 16, still give the same final numerical answer as the expressions in qu. #1.

a) $\sum_{i=1}^{5}3i+\sum_{i=1}^{5}1$ b) $3\sum_{i=1}^{5}i+\sum_{i=1}^{5}1$