# Infinite Geometric Series (notes)

**Date:**

Consider an infinite geometric series.

Ex. $S_\infty = 3 + 3(2) + 3(2^2) + 3(2^3) + \ldots$
$= 3 + 6 + 12 + 24 + \ldots$

Clearly, the sum keeps getting larger. Each subsequent term is larger, so the sum keeps getting larger. The sum is said to ***approach infinity*** or ***diverge***.

When the common ratio is less than one, each subsequent term is smaller, and the sum can ***converge***. This means that a decreasing series can ***sum to a finite amount***. Here, we will examine an infinite geometric series. An infinite geometric series converges when the common ratio $-1 < r < 1$.

Deriving a formula for the sum of an infinite geometric series. Consider an infinite geometric series:

$$S_\infty = a + ar + ar^2 + ar^3 + \ldots$$

$$rS_\infty = ar + ar^2 + ar^3 + ar^4 \ldots \qquad \text{(multiplying the sum by } r\text{)}$$

$$S_\infty - rS_\infty = a + 0 + 0 + 0 + \ldots \qquad \text{(subtracting the two equations)}$$

$$S_\infty(1 - r) = a \qquad \text{(factoring...)}$$

Thus, $S_\infty = \dfrac{a}{1-r}, \ -1 < r < 1$

Ex. Evaluate the following infinite series from first principles using sigma notation:

a) $0.\overline{3} = 0.3 + 0.03 + 0.003 + \ldots$

$a = 0.3, r = 0.1$

*and* $S_\infty = \dfrac{a}{1-r}$

*So* $0.\overline{3} = \dfrac{0.3}{1-0.1} = \dfrac{0.3}{0.9} = \dfrac{1}{3}$, as expected.

b) $2 + 2\left(\dfrac{1}{3}\right) + 2\left(\dfrac{1}{3}\right)^2 + \ldots =$

$a = 2, r = \dfrac{1}{3}$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{2}{1-\frac{1}{3}} \\ &= \frac{2}{\left(\frac{2}{3}\right)} \\ &= 3 \end{aligned}$$

$\therefore 2 + \dfrac{2}{3} + \dfrac{2}{9} + \dfrac{2}{27} + \ldots = 3$ (*Wow!*)