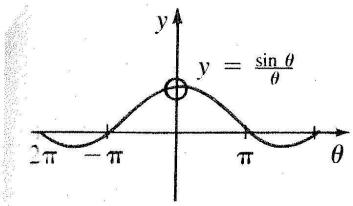
Limit Of Trig



Approaching 0 From the Right

θ	$\frac{\sin\theta}{\theta}$
0.3	0.985 067
0.2	0.993 347
0.1	0.998 334
0.05	0.999 583
0.02	0.999 933
0.01	0.999 983

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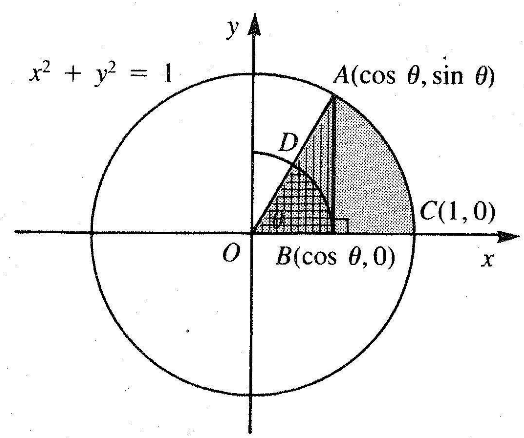
The trend of the values of $\frac{\sin\theta}{\theta}$ in the tables suggests that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. A proof follows.

In the diagram, point A is on the unit circle $x^2 + y^2 = 1$. Point A determines angle θ, $0 < \theta < \frac{\pi}{2}$. The perpendicular drawn from point A meets the x-axis at B. The circle, radius OB, meets line segment OA at D.



Sector area $= \frac{1}{2}r^2\theta$

Area of Triangle $= \frac{1}{2}bh$

Now Area sector $OBD \leqslant$ Area triangle $OAB \leqslant$ Area sector OAC

Therefore $\frac{1}{2}(OB)^2(\theta) \leqslant \frac{1}{2}(OB)(BA) \leqslant \frac{1}{2}(OC)^2(\theta)$

$$\left(\frac{1}{2}\cos^2\theta\right)(\theta) \leqslant \frac{1}{2}\cos\theta\sin\theta \leqslant \frac{1}{2}(1)^2(\theta)$$

$$\theta\cos^2\theta \leqslant \cos\theta\sin\theta \leqslant \theta$$

$$\frac{\theta\cos^2\theta}{\theta\cos\theta} \leqslant \frac{\cos\theta\sin\theta}{\theta\cos\theta} \leqslant \frac{\theta}{\theta\cos\theta}, \quad \theta\cos\theta > 0$$

$$\cos\theta \leqslant \frac{\sin\theta}{\theta} \leqslant \frac{1}{\cos\theta}$$

Applying the squeeze or sandwich theorem.

and $\lim_{\theta\to 0^+}\cos\theta \leqslant \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} \leqslant \lim_{\theta\to 0^+}\frac{1}{\cos\theta}$

Therefore $1 \leqslant \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} \leqslant 1$

and $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$

Since $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function, $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$.

Therefore $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$ ①