Deriving the Taylor and Maclaurin Series Formulas

Date:

Recall: Power Series

An expression of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ...$ is a **power series** centered at ***x* = *a*.** The term $c_n(x-a)^n$ is the *n*th term; the number *a* is the center.

An expression of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + ... + c_nx^n + ...$ is a **power series** centered at ***x* = 0 (Maclaurin Series)**

Deriving the Taylor Expansion Formula:

Let's assume a function $f(x)$ has a power series representation so that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + ... + c_n(x-a)^n + ... \quad \text{centered at } x = a$$

Notice 1:

$f(a) = c_0 + c_1(0) + c_2(0) + ...$

$f(a) = c_0$

$$\boxed{c_0 = f(a)}$$

Notice 2:

$\because f'(x) = c_1(1) + 2c_2(x-a)^1 + 3c_3(x-a)^2 + 4c_4(x-a)^3 + ...$

$\therefore f'(a) = c_1 + 2c_2(0) + 3c_3(0) + 4c_4(0) + ...$

$f'(a) = c_1$

$$\boxed{c_1 = f'(a)}$$

Notice 3:

$\because f''(x) = (2)(1)c_2(1) + (3)(2)c_3(x-a)^1 + (4)(3)c_4(x-a)^2 + ...$

$\therefore f''(a) = (2)(1)c_2 + (3)(2)c_3(0) + (4)(3)c_4(0) + ...$

$f''(a) = (2)(1)c_2$

$$\boxed{c_2 = \frac{f''(a)}{(2)(1)} \to \frac{f''(a)}{2!}}$$

Notice 4:

$\because f'''(x) = (3)(2)(1)c_3(1) + (4)(3)(2)c_4(x-a)^1 + ...$

$\therefore f'''(a) = (3)(2)(1)c_3 + (4)(3)(2)c_4(0) + ...$

$f'''(a) = (3)(2)(1)c_3$

$$\boxed{c_3 = \frac{f'''(a)}{(3)(2)(1)} \to \frac{f'''(a)}{3!}}$$

....

If the procedures going to continue,

$f^{(4)}(a) = (4)(3)(2)(1)c_4$

$$\boxed{c_4 = \frac{f^{(4)}(a)}{4!}}$$

In general:

$f^{(n)}(a) = n!c_n$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Recall

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+\ldots+c_n(x-a)^n+\ldots$$

Now, let`s substitute the c values into it to form the Taylor series expansions

$$\begin{aligned}f(x)&=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots\\&=\frac{f(a)}{0!}(x-a)^0+\frac{f'(a)}{1!}(x-a)^1+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots\end{aligned}$$

or

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall: $0!=1$

Taylor Series Expansions (Centered at $x=a$)

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Series Expansions (Centered at $x=0$)

$$f(x)=f(0)+f'(0)(x)+\frac{f''(0)}{2!}(x)^2+\frac{f'''(0)}{3!}(x)^3+\frac{f^{(4)}(0)}{4!}(x)^4+\ldots+\frac{f^{(n)}(0)}{n!}(x)^n+\ldots$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x)^n$$