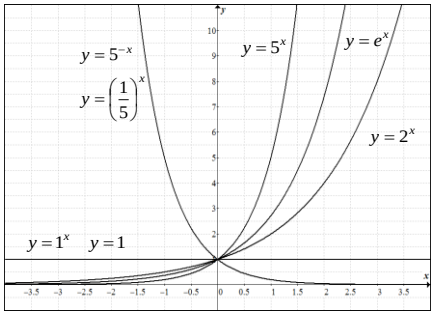
The Natural Exponential & Logarithmic Functions

Date:

Definition of *e* (Natural Exponential Number)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \approx 2.718281828459.....$$



Properties of $y = e^x$ and $y = \ln x$

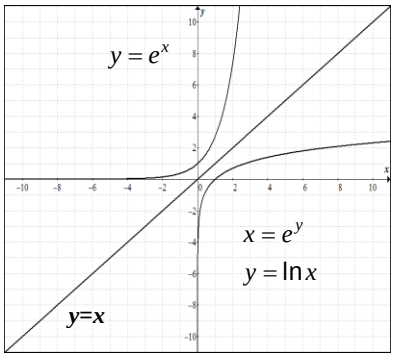
Recall the logarithmic function is the inverse of the exponential function.

$\because y = \log_b x \Leftrightarrow b^y = x$ which is the inverse of $b^x = y$,

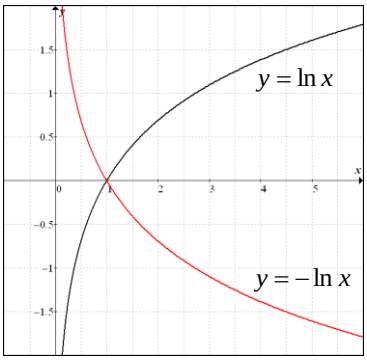
$\therefore y = \log_e x \Leftrightarrow e^y = x$ which is the inverse of $y = e^x$.

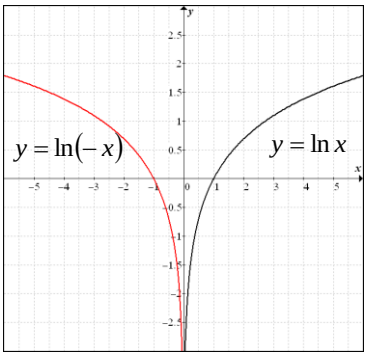
The function $y = \log_e x$ can be written as $y = \ln x$ and is called the natural logarithm function.

$$y = \log_b x \Leftrightarrow b^y = x$$
$$y = \ln x \Leftrightarrow e^y = x$$



$y = e^x$	$y = \ln x$
• **Domain is** $x \in R$	• **Domain is** $x \in R \mid x > 0$
• **Range is** $y \in R \mid y > 0$	• **Range is** $y \in R$
• ***y*-intercept at 1**	• ***x*-intercept at 1**
• $e^{\ln x} = x, x > 0$	• $\ln e^x = x, x \in R$
• **Horizontal Asymptote** $y = 0$.	• **Vertical Asymptote** $x = 0$.





Basic Properties

1) $\log_e 1 = 0 \quad \Leftrightarrow \ln 1 = 0$
2) $\log_e e = 1 \quad \Leftrightarrow \ln e = 1$
3) $\log_e e^x = x \quad \Leftrightarrow \ln e^x = x,\ x \in R$
4) $e^{\log_e x} = x \quad \Leftrightarrow e^{\ln x} = x,\ x > 0$

Definition of Natural Logarithm

Natural logarithm is the logarithm to the base *e*. $\log_e x \Leftrightarrow \ln x$

Laws of Logarithm

1) $\log_e xy = \log_e x + \log_e y \quad \Leftrightarrow \ln xy = \ln x + \ln y$ (Product Law)
2) $\log_e \frac{x}{y} = \log_e x - \log_e y \quad \Leftrightarrow \ln \frac{x}{y} = \ln x - \ln y$ (Quotient Law)
3) $\log_e x^p = p \log_e x \quad \Leftrightarrow \ln x^p = p \ln x$ (Power Law)

Example 1: Natural Logarithms

Solve for x: $\ln x = 8$

Example 2: Solving Exponential Equations

Solve the equation $e^{3-2x} = 4$

Example 3: Natural Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

Example 4: Single Natural Logarithm

Express $\frac{1}{2}\ln x - 4\ln y + \ln(x^2 + 1)$ as a single logarithm.

Exponential Growth & Decay

If $y = f(t)$ is the number of individuals in a population of animals or bacteria cells at time *t*, then the size of the population at time *t* is $y = y_0 e^{kt}$.

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Example 5: Exponential Growth

A bacteria culture starts with 2000 bacteria and after 3 h the estimates count is 10 000 bacteria.

a) Find the number of bacteria after *t* hours.
b) Find the number of bacteria after 2 hours.
c) When will the bacteria population reach 18 000?

Example 6: Exponential Growth

The half-life of Polonium-210 is 140 days and a sample of this element has a mass of 300 mg.

a) Find the mass that remains after *t* days.
b) Find the mass that remains after 50 days.
c) How long will the sample take to decay to a mass of 200 mg?

HW: Worksheet
Natural Logarithmic Exercise

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Exercise

1. Graph each function, state the domain, range, and asymptote of each function.
   a) $g(x) = \ln(x+2)$ b) $y = 1 - \ln x$ c) $y = 1 + \ln(-x)$
   d) $y = |\ln x|$ e) $y = \ln|x|$

2. Evaluate without using a calculator.
   a) $e^{\ln 5}$ b) $\ln e^2$ c) $2\ln e$ d) $e^{5\ln 2}$ e) $\ln\sqrt{e}$ f) $\ln 2 + 2\ln 3 - \ln 18$

3. Solve for *x*:
   a) $e^x = 4$ b) $\ln x = 6$ c) $\ln(2x-1) = 1$ d) $e^{3x+5} = 10$ e) $\ln\left(e^{3-x}\right) = 8$
   f) $\ln x = \ln 4 + \ln 7$ g) $\ln(\ln x) = 2$ h) $e^{e^x} = 5$

4. Find the solution of each equation correct to six decimal places.
   a) $\ln(x+1) = 3$ b) $e^{-x} = \frac{1}{2}$ c) $e^{5x+3} = 10$ d) $2^{x-5} = 3$

5. Express as a single logarithm.
   a) $\frac{1}{3}\ln x + 2\ln(3x-5)$ b) $2\ln x - \frac{1}{2}\ln\left(x^2-1\right) + 3\ln\left(x^2+1\right)$

6. Find the domain of each function.
   a) $g(x) = \ln\left(x - x^2\right)$ b) $h(x) = \ln x - \ln(2-x)$ c) $h(x) = \sqrt{x-2} - \ln(10-x)$

7. A bacteria culture starts with 1000 bacteria. After 2 h the estimated count is 10000 bacteria.
   a) Find the number of bacteria after *t* hours.
   b) Find the number of bacteria after 5 h.
   c) When will the bacteria population reach 15000?

8. The initial size of a bacteria culture is 400. After an hour there are 1200 bacteria.
   a) Find the number of bacteria after *t* hours.
   b) In what period of time does the population double?

9. A cell of the bacterium Escherichia coli in a nutrient broth medium divides into two cells every 20 min. Suppose that there are initially 500 cells. Find
   a) the number of cells after *t* hours.
   b) the number of cells after 8 hours.
   c) the time required for the size to reach 6000 cells.

10. The population of the world is doubling about every 35 years. In 1987 the population reached 5 billion.
    a) Find the projected world population for the year 2100.
    b) When will the world population reach 50 billion?

11. Uranium-238 has a half of 4.5×10^9 years.
    a) Find the mass that remains from a 100 mg sample after *t* years.
    b) Find the mass that remains from this sample after 10000 years.

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Answers

1a) $D:(-2,\infty)$ $R: y \in R$, Asymp: $x=-2$ b) $D:(0,\infty)$ $R: y \in R$, Asymp: $x=0$

c) $D:(-\infty,0)$ $R: y \in R$, Asymp: $x=0$ d) $D:(0,\infty)$ $R:(0,\infty)$, Asymp: $x=0$

e) $D:(-\infty,0)\cup(0,\infty)$ $R: y \in R$, Asymp: $x=0$

2a) 5 b) 2 c) 2 d) 32 e) 0.5 f) 0

3a) ln4 b) e^6 c) 0.5(e+1) d) $\frac{1}{3}(\ln 10-5)$ e) -5 f) 28 g) e^{e^2} h) $\ln(\ln 5)$

4a) 19.085537 b) 0.693147 c) -0.139483 d) 6.584963

5a) $\ln\left[\sqrt[3]{x}(3x-5)^2\right]$ b) $\ln\left(\frac{x^2(x^2+1)^3}{\sqrt{x^2-1}}\right)$

6a) (0, 1) b) (0, 2) c) [2, 10)

7a) $(1000)10^{\frac{t}{2}}$ b) 316228 c) 2 hr 21 min.

8a) $(400)3^t$ b) $\frac{\ln 2}{\ln 3}h \approx 37.5\,min$

9a) $(500)2^{3t}$ b) 8.389 x 10^9 c) $\frac{\ln 12}{3\ln 2}h \approx 1hr\,11min$

10a) 47 billion b) 2103

11a) $(100)2^{-\frac{t}{4.5\times10^9}}$ b) 99.999846 mg