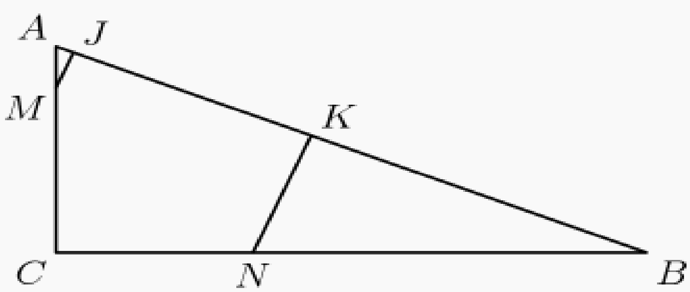
1.

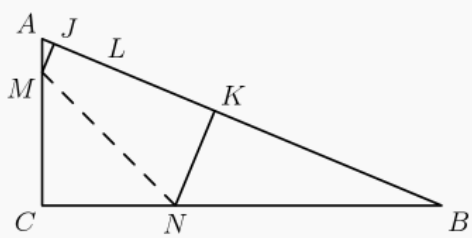
In ABC, AB=13, and BC=12. Points M and N lie on AC and BC, respectively, with CM=CN=4. Points J and K are on AB so that MJ and NK are perpendicular to AB. What is the area of pentagon CMJKN?



Multiply your answer by 130 then minus 311, go to that locker

Answer:

Split the pentagon along a different diagonal as follows:



The area of the pentagon is then the sum of the areas of the resulting right triangle and trapezoid. As before, triangles ABC, AMJ, and NBK are all similar.

Since $BN = 12 - 4 = 8$, $NK = \frac{5}{13}(8) = \frac{40}{13}$ and $BK = \frac{12}{13}(8) = \frac{96}{13}$. Since $AM = 5 - 4 = 1$, $JM = \frac{12}{13}$ and $AJ = \frac{5}{13}$.

The trapezoid's height is therefore $13 - \frac{5}{13} - \frac{96}{13} = \frac{68}{13}$, and its area is $\frac{1}{2}\left(\frac{68}{13}\right)\left(\frac{12}{13} + \frac{40}{13}\right) = \frac{34}{13}(4) = \frac{136}{13}$.

Triangle MCN has area $\frac{1}{2}(4)(4) = 8$, and the total area is $\frac{104 + 136}{13} = \boxed{\frac{240}{13}}$.

2.

Several sets of prime numbers, such as {7,83,421,659} use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?

Multiply your answer by 11 then minus 1210, go to that locker

Answer:

Neither of the digits 4, 6, and 8 can be a unit's digit of a prime. Therefore the sum of the set is at least 40 + 60 + 80 + 1 + 2 + 3 + 5 + 7 + 9 = 207.

We can indeed create a set of primes with this sum, for example the following set works: { 41, 67, 89, 2, 3, 5 }

Thus the answer is 207

3.

Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal 0.ababab... is expressed as a fraction in lowest terms. How many different denominators are possible?

Multiply your answer by 400 then plus 8, go to that room

Answer:

The repeating decimal $0.\overline{ab}$ is equal to

$$\frac{10a+b}{100}+\frac{10a+b}{10000}+\cdots=(10a+b)\cdot\left(\frac{1}{10^2}+\frac{1}{10^4}+\cdots\right)=(10a+b)\cdot\frac{1}{99}=\frac{10a+b}{99}$$

When expressed in lowest terms, the denominator of this fraction will always be a divisor of the number $99=3\cdot3\cdot11$. This gives us the possibilities $\{1,3,9,11,33,99\}$. As a and b are not both nine and not both zero, the denominator 1 can not be achieved, leaving us with $\boxed{(C)5}$ possible denominators.

(The other ones are achieved e.g. for ab equal to 33, 11, 9, 3, and 1, respectively.)

4

The director of a marching band wishes to place the members into a formation that includes all of them and has no unfilled positions. If they are arranged in a square formation, there are 5 members left over. The director realizes that if he arranges the group in a formation with 7 more rows than columns, there are no members left over. Find the maximum number of members this band can have.

Multiply your answer by 5 then plus 42, go to that locker

Answer:

Define the number of rows/columns of the square formation as s, and the number of rows of the rectangular formation r (so there are $r-7$ columns). Thus, $s^2+5=r(r-7)\implies r^2-7r-s^2-5=0$. The quadratic formula yields $r=\frac{7\pm\sqrt{49-4(1)(-s^2-5)}}{2}=\frac{7\pm\sqrt{4s^2+69}}{2}$. $\sqrt{4s^2+69}$ must be an integer, say x. Then $4s^2+69=x^2$ and $(x+2s)(x-2s)=69$. The factors of 69 are $(1,69)$, $(3,23)$; x is maximized for the first case. Thus, $x=\frac{69+1}{2}=35$, and $r=\frac{7\pm35}{2}=21,-14$. The latter obviously can be discarded, so there are 21 rows and $21-7=14$ columns, making the answer 294.

5.

Let m be the number of five-element subsets that can be chose from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive.

Plus your answer by 274, go to that room

Answer:

Let A be the number of ways in which 5 distinct numbers can be selected from the set of the first 14 natural numbers, and let B be the number of ways in which 5 distinct numbers, no two of which are consecutive, can be selected from the same set. Then $m = A - B$. Because $A = \binom{14}{5}$, the problem is reduced to finding B. Consider the natural numbers $1 \leq a_1 < a_2 < a_3 < a_4 < a_5 \leq 14$. If no two of them are consecutive, the numbers $b_1 = a_1, b_2 = a_2 - 1, b_3 = a_3 - 2, b_4 = a_4 - 3$, and $b_5 = a_5 - 4$ are distinct numbers from the interval $[1, 10]$. Conversely, if $b_1 < b_2 < b_3 < b_4 < b_5$ are distinct natural numbers from the interval $[1, 10]$, then $a_1 = b_1$, $a_2 = b_2 + 1$, $a_3 = b_3 + 2$, $a_4 = b_4 + 3$, and $a_5 = b_5 + 4$ are from the interval $[1, 14]$, and no two of them are consecutive. Therefore counting B is the same as counting the number of ways of choosing 5 distinct numbers from the set of the first 10 natural numbers. Thus $B = \binom{10}{5}$. Hence $m = A - B = \binom{14}{5} - \binom{10}{5} = 2002 - 252 = 1750$ and the answer is 750.