2013 3 In the Fibonacci sequence, 1; 1; 2; 3; 5; ----- each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

Notice there's a pattern of odd odd even odd odd even……

Odd+odd=even & odd +even = odd, therefore the pattern will continue

So for the first 99 terms there are 66 odds, and the 100^{th} term is also odd,

So there are 67 odds in the first 100 terms

2016 6 Determine all possible values for the area of a right-angled triangle with one side length equal to 60 and with the property that its side lengths form an arithmetic sequence.

Suppose that a triangle has side lengths in arithemetic sequence.
Then the side lengths can be written as $a - d, a, a + d$ for some $a > 0$ and $d \geq 0$.
Note that $a - d \leq a \leq a + d$.
For such a triangle to be right-angled, by the Pythagorean Theorem, the following equivalent equations are true:

$$(a-d)^2 + a^2 = (a+d)^2$$
$$a^2 - 2ad + d^2 + a^2 = a^2 + 2ad + d^2$$
$$a^2 = 4ad$$

Since $a > 0$, then $a = 4d$, and so the side lengths of the triangle are $a - d = 3d$, $a = 4d$, and $a + d = 5d$ for some $d \geq 0$.
(Note that such triangles are all similar to the 3-4-5 triangle.)
If such a triangle has 60 as a side length, then there are three possibilities:

(i) $3d = 60$: This gives $d = 20$ and side lengths $60, 80, 100$.
Since the triangle is right-angled and its hypotenuse has length 100, then its area will equal $\frac{1}{2}(60)(80) = 2400$.

(ii) $4d = 60$: This gives $d = 15$ and side lengths $45, 60, 75$.
In a similar way to case (i), its area will equal $\frac{1}{2}(45)(60) = 1350$.

(iii) $5d = 60$: This gives $d = 12$ and side lengths $36, 48, 60$.
In a similar way to case (i), its area will equal $\frac{1}{2}(36)(48) = 864$.

Therefore, the possible values for the area of such a triangle are 2400, 1350, and 864.

2014 6 The geometric sequence with n terms t_1; t_2; ----- t_{n-1}; t_n has $t_1 t_n = 3$. Also, the product of all n terms equals 59 049 (that is, $t_1*t_2*....*t_{n-1}*t_n = 59\,049$). Determine the value of n.

(b) Suppose that the first term in the geometric sequence is $t_1 = a$ and the common ratio in the sequence is r.
Then the sequence, which has n terms, is $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$.
In general, the kth term is $t_k = ar^{k-1}$; in particular, the nth term is $t_n = ar^{n-1}$.
Since $t_1t_n = 3$, then $a \cdot ar^{n-1} = 3$ or $a^2r^{n-1} = 3$.
Since $t_1t_2 \cdots t_{n-1}t_n = 59\,049$, then

$$\begin{aligned}(a)(ar)\cdots(ar^{n-2})(ar^{n-1}) &= 59\,049 \\ a^n rr^2 \cdots r^{n-2}r^{n-1} &= 59\,049 \quad \text{(since there are } n \text{ factors of } a \text{ on the left side)} \\ a^n r^{1+2+\cdots+(n-2)+(n-1)} &= 59\,049 \\ a^n r^{\frac{1}{2}(n-1)(n)} &= 59\,049\end{aligned}$$

since $1 + 2 + \cdots + (n-2) + (n-1) = \frac{1}{2}(n-1)(n)$.
Since $a^2r^{n-1} = 3$, then $(a^2r^{n-1})^n = 3^n$ or $a^{2n}r^{(n-1)(n)} = 3^n$.
Since $a^n r^{\frac{1}{2}(n-1)(n)} = 59\,049$, then $\left(a^n r^{\frac{1}{2}(n-1)(n)}\right)^2 = 59\,049^2$ or $a^{2n}r^{(n-1)(n)} = 59\,049^2$.
Since the left sides of these equations are the same, then $3^n = 59\,049^2$.
Now

$$59\,049 = 3(19\,683) = 3^2(6561) = 3^3(2187) = 3^4(729) = 3^5(243) = 3^6(81) = 3^6 3^4 = 3^{10}$$

Since $59\,049 = 3^{10}$, then $59\,049^2 = 3^{20}$ and so $3^n = 3^{20}$, which gives $n = 20$.

2015 7 The numbers a_1; a_2; a_3; …… form an arithmetic sequence with $a_1 \neq a_2$. The three numbers a_1; a_2; a_6 form a geometric sequence in that order. Determine all possible positive integers k for which the three numbers a_1; a_4; a_k also form a geometric sequence in that order.

Suppose that the arithmetic sequence $a_1, a_2, a_3, \ldots$ has first term a and common difference d.
Then, for each positive integer n, $a_n = a + (n-1)d$.
Since $a_1 = a$ and $a_2 = a + d$ and $a_1 \neq a_2$, then $d \neq 0$.
Since a_1, a_2, a_6 form a geometric sequence in that order, then $\frac{a_2}{a_1} = \frac{a_6}{a_2}$ or $(a_2)^2 = a_1 a_6$.
Substituting, we obtain

$$\begin{aligned} (a+d)^2 &= a(a+5d) \\ a^2 + 2ad + d^2 &= a^2 + 5ad \\ d^2 &= 3ad \\ d &= 3a \qquad (\text{since } d \neq 0) \end{aligned}$$

Therefore, $a_n = a + (n-1)d = a + (n-1)(3a) = (3n-2)a$ for each $n \geq 1$.
Thus, $a_4 = (3(4)-2)a = 10a$, and $a_k = (3k-2)a$. (Note that $a_1 = (3(1)-2)a = a$.)
For a_1, a_4, a_k to also form a geometric sequence then, as above, $(a_4)^2 = a_1 a_k$, and so

$$\begin{aligned} (10a)^2 &= (a)((3k-2)a) \\ 100a^2 &= (3k-2)a^2 \end{aligned}$$

Since $d \neq 0$ and $d = 3a$, then $a \neq 0$.
Since $100a^2 = (3k-2)a^2$ and $a \neq 0$, then $100 = 3k - 2$ and so $3k = 102$ or $k = 34$.
Checking, we note that $a_1 = a$, $a_4 = 10a$ and $a_{34} = 100a$ which form a geometric sequence with common ratio 10.
Therefore, the only possible value of k is $k = 34$.

2012 10 For each positive integer N, an Eden sequence from {1; 2; 3; …; N} is defined to be a sequence that satisfies the following conditions:

(i) each of its terms is an element of the set of consecutive integers {1; 2; 3; … ;N},

(ii) the sequence is increasing, and

(iii) the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from {1; 2; 3} are

1 3 1; 2 1; 2; 3

(a) Determine the number of Eden sequences from {1; 2; 3; 4; 5}.

(b) For each positive integer N, define e(N) to be the number of Eden sequences from {1; 2; 3; …. ;N}. If e(17) = 4180 and e(20) = 17710, determine e(18) and e(19).

a. 1, 3, 5, 12, 14, 34, 123, 125, 145, 345, 1234, 12345. ∴ $e(5) = 12$

b. let $O(N)$ be number of eden sequence that ends with an odd number from $\{1, 2, \cdots N\}$

.. $E(N)$ . . . . even . . .

∴ $e(N) = O(N) + E(N)$

notice when N is odd,

$O(N) = O(N-1) + E(N-1) + 1$

because from set $\{1 \cdots N-1\}$, then Edens that ends in odd number can also be chosen from $\{1 \cdots N\}$, while ones ends in even number can form a new Eden by adding number N in it's end, and there's also a new sequence N

Also, $E(N) = E(N-1)$

because no new terms that ends in even are formed with the additional choice of N (since N is odd)

Similarly

when N is Even

$O(N) = O(N-1)$

$E(N) = E(N-1) + O(N-1)$

notice there's no +1 this time because N is no longer an Eden

with this pattern,

$e(17) = E(17) + O(17)$

$E(18) = E(17) + O(17)$ $\quad O(18) = O(17)$, $\boxed{e(18) = 2(O(17)) + E(17)}$ ①

$E(19) = E(18) = E(17) + O(17)$ $\quad O(19) = E(18) + O(18) + 1 = 2(O(17)) + E(17) + 1$

$\boxed{e(19) = E(19) + O(19) = 3(O(17)) + 2(E(17)) + 1}$ ②

$e(20) = E(20) + O(20) = (E(19) + O(19)) + (O(19)) = 5(O(17)) + 3(E(17)) + 2$

∴ $\begin{cases} E(17) + O(17) = 4180 \\ 5(O(17)) + 3(E(17)) + 2 = 17710 \end{cases}$

Solve the equation and sub answers into ① & ②

$e(18) = 6764$

$e(19) = 10945$

* challenge : try to prove $e(N) + e(N+1) = e(N+2)$ & solve the problem