THE MATHEMATICAL ASSOCIATION OF AMERICA
**American Mathematics Competitions**

60th Annual American Mathematics Contest 12

# AMC 12
# CONTEST A

## Solutions Pamphlet

Tuesday, FEBRUARY 10, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*



*Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:*

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

Prof. Bernardo Abrego, Dept. of Mathematics,
California State University-Northridge, Northridge, CA 91330-8313

1. **Answer (A):** There are $60-34=26$ minutes from 10:34 AM to 11:00 AM, there are 2 hours from 11:00 AM to 1:00 PM, and there are 18 minutes from 1:00 PM to 1:18 PM. Thus the flight lasted 2 hours and $26+18=44$ minutes. Hence $h+m=2+44=46$.

2. **Answer (C):** Simplifying the expression,

$$1+\frac{1}{1+\frac{1}{1+1}}=1+\frac{1}{1+\frac{1}{2}}=1+\frac{1}{\frac{3}{2}}=1+\frac{2}{3}=\frac{5}{3}.$$

3. **Answer (B):** The number is

$$\frac{1}{4}+\frac{1}{3}\left(\frac{3}{4}-\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}.$$

4. **Answer (A):** The value of any combination of four coins that includes pennies cannot be a multiple of 5 cents, and the value of any combination of four coins that does not include pennies must exceed 15 cents. Therefore the total value cannot be 15 cents. The other four amounts can be made with, respectively, one dime and three nickels; three dimes and one nickel; one quarter, one dime and two nickels; and one quarter and three dimes.

5. **Answer (D):** Let $x$ be the side length of the cube. Then the volume of the cube was $x^3$, and the volume of the new solid is $x(x+1)(x-1)=x^3-x$. Therefore $x^3-x=x^3-5$, from which $x=5$, and the volume of the cube was $5^3=125$.

6. **Answer (E):** Note that

$$12^{mn}=(2^2\cdot 3)^{mn}=2^{2mn}\cdot 3^{mn}=(2^m)^{2n}\cdot(3^n)^m=P^{2n}Q^m.$$

Remark: The pair of integers $(2,1)$ shows that the other choices are not possible.

7. **Answer (B):** Because the difference between consecutive terms is constant,

$$(5x-11)-(2x-3)=(3x+1)-(5x-11).$$

Therefore $x=4$, and the first three terms are 5, 9, and 13. Thus the difference between consecutive terms is 4. The $n$th term is $2009=5+(n-1)\cdot 4$, and it follows that $n=502$.

8. **Answer (A):** Let the lengths of the shorter and longer side of each rectangle be $x$ and $y$, respectively. The outer and inner squares have side lengths $y + x$ and $y - x$, respectively, and the ratio of their side lengths is $\sqrt{4} = 2$. Therefore $y + x = 2(y - x)$, from which $y = 3x$.

9. **Answer (D):** Expanding, we have $f(x+3) = a(x^2 + 6x + 9) + b(x+3) + c = ax^2+(6a+b)x+(9a+3b+c)$. Equating coefficients implies that $a = 3$, $6\cdot 3+b = 7$, whence $b = -11$, and then $9 \cdot 3 + 3 \cdot (-11) + c = 4$, and so $c = 10$. Therefore $a + b + c = 3 - 11 + 10 = 2$.

**OR**

Note that

$$\begin{aligned} f(x) = f((x-3)+3) &= 3(x-3)^2 + 7(x-3) + 4 \\ &= 3(x^2 - 6x + 9) + 7x - 21 + 4 \\ &= 3x^2 - 11x + 10. \end{aligned}$$

Therefore $a = 3$, $b = -11$, and $c = 10$, giving $a + b + c = 2$.

**OR**

The sum $a + b + c$ is $f(1) = f(-2 + 3) = 3(-2)^2 + 7(-2) + 4 = 2$.

10. **Answer (C):** Let $x$ be the length of $\overline{BD}$. By the triangle inequality on $\triangle BCD$, $5 + x > 17$, so $x > 12$. By the triangle inequality on $\triangle ABD$, $5 + 9 > x$, so $x < 14$. Since $x$ must be an integer, $x = 13$.

11. **Answer (E):** The outside square for $F_n$ has 4 more diamonds on its boundary than the outside square for $F_{n-1}$. Because the outside square of $F_2$ has 4 diamonds, the outside square of $F_n$ has $4(n-2)+4 = 4(n-1)$ diamonds. Hence the number of diamonds in figure $F_n$ is the number of diamonds in $F_{n-1}$ plus $4(n-1)$, or

$$\begin{aligned} & 1 + 4 + 8 + 12 + \cdots + 4(n-2) + 4(n-1) \\ =\, & 1 + 4(1 + 2 + 3 + \cdots + (n-2) + (n-1)) \\ =\, & 1 + 4\frac{(n-1)n}{2} \\ =\, & 1 + 2(n-1)n. \end{aligned}$$

Therefore figure $F_{20}$ has $1 + 2 \cdot 19 \cdot 20 = 761$ diamonds.

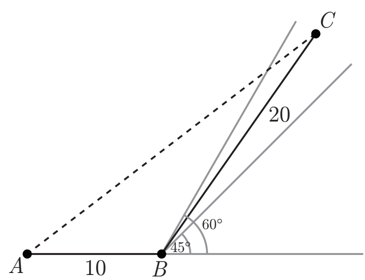
12. **Answer (B):** The only such number is 54. A single-digit number would have to satisfy $6u = u$, implying $u = 0$, which is impossible. A two-digit number would have to satisfy $10t + u = 6(t + u)$, so $4t = 5u$ and then necessarily $t = 5$ and $u = 4$; hence the number is 54. A three-digit number would have to satisfy $100h+10t+u = 6(h+t+u)$ or $94h+4t = 5u$. But the left side of the expression is at least 94 while the right side of the expression is at most 45, so no solution is possible.

13. **Answer (D):** By the Law of Cosines,

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\angle ABC = 500 - 400\cos\angle ABC.$$

Because $\cos\angle ABC$ is between $\cos 120° = -\frac{1}{2}$ and $\cos 135° = -\frac{\sqrt{2}}{2}$, it follows that

$$700 = 500 + 200 \le AC^2 \le 500 + 200\sqrt{2} < 800.$$




14. **Answer (B):** The line must contain the midpoint of the segment joining $(1, 1)$ and $(6m, 0)$, which is $\left(\frac{6m+1}{2}, \frac{1}{2}\right)$. Thus

$$m = \frac{\frac{1}{2}}{\frac{6m+1}{2}} = \frac{1}{6m+1},$$

from which $0 = 6m^2 + m - 1 = (3m - 1)(2m + 1)$. The two possible values of $m$ are $-\frac{1}{2}$ and $\frac{1}{3}$, and their sum is $-\frac{1}{6}$.

If $m = -\frac{1}{2}$ then the triangle with vertices $(0, 0)$, $(1, 1)$, and $(-3, 0)$ is bisected by the line passing through the origin and $(-1, \frac{1}{2})$. Similarly, when $m = \frac{1}{3}$ the triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$ is bisected by the line passing through the origin and $(\frac{3}{2}, \frac{1}{2})$.

15. **Answer (D):** Let $k$ be a multiple of 4. For $k \geq 0$,

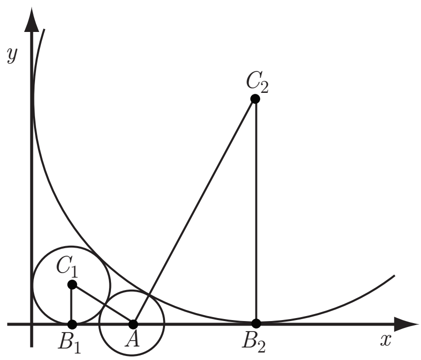
$$(k+1)\,i^{k+1} + (k+2)\,i^{k+2} + (k+3)\,i^{k+3} + (k+4)\,i^{k+4} = \\ (k+1)\,i + (k+2)\,(-1) + (k+3)\,(-i) + (k+4) = 2 - 2i.$$

Thus when $n = 4 \cdot 24 = 96$, we have $i + 2i^2 + \cdots + ni^n = 24\,(2-2i) = 48 - 48i$. Adding the term $97i^{97} = 97i$ gives $(48-48i) + 97i = 48 + 49i$ when $n = 97$.

16. **Answer (D):** Let $r$ be the radius of a circle with center $C$, $A = (3,0)$, and $B = (r,0)$. Then, $AC = 1+r$ and $CB = r$. Applying the Pythagorean Theorem to $\triangle ABC$ gives

$$AB^2 = (1+r)^2 - r^2 = 1 + 2r.$$

Also, $AB = |3-r|$, so $1 + 2r = (3-r)^2$, which simplifies to $r^2 - 8r + 8 = 0$. Thus $r = 4 \pm 2\sqrt{2}$, both of which are positive, and the sum of all possible values of $r$ is 8.




17. **Answer (C):** The sum of the first series is

$$\frac{a}{1-r_1} = r_1,$$

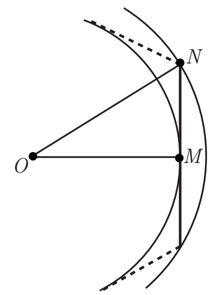
from which $r_1^2 - r_1 + a = 0$, and $r_1 = \frac{1}{2}(1 \pm \sqrt{1-4a})$. Similarly, $r_2 = \frac{1}{2}(1 \pm \sqrt{1-4a})$. Because $r_1$ and $r_2$ must be different, $r_1 + r_2 = 1$. Such series exist as long as $0 < a < \frac{1}{4}$.

18. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by $2^6$, so $N(k) < 6$. For $k > 4$, the first term is divisible by $2^7$, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6+1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6+1) = ((5^2)^3 + 1^3) = (5^2+1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2+1=26$ is divisible by 2 but not 4, and the second factor is odd, so $5^6+1$ contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

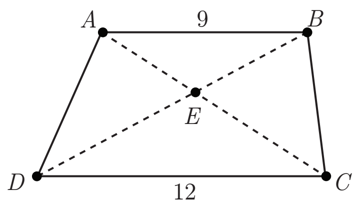
19. **Answer (C):** Consider a regular $n$-gon with side length 2. Let the radii of its inscribed and circumscribed circles be $r$ and $R$, respectively. Let $O$ be the common center of the circles, let $M$ be the midpoint of one side of the polygon, and let $N$ be one endpoint of that side. Then $\triangle OMN$ has a right angle at $M$, $MN=1$, $OM=r$, and $ON=R$. By the Pythagorean Theorem, $R^2-r^2=1$. Thus the area of the annulus between the circles is $\pi(R^2-r^2)=\pi$ for all $n\geq 3$. Hence $A=B$.




20. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from $A$ and $B$ to $\overline{CD}$ have the same length. Thus $\overline{AB}\parallel\overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

$$\frac{AE}{EC}=\frac{AB}{CD}=\frac{9}{12}=\frac{3}{4}.$$

Let $AE=3x$ and $EC=4x$. Then $7x=AE+EC=AC=14$, so $x=2$, and $AE=3x=6$.

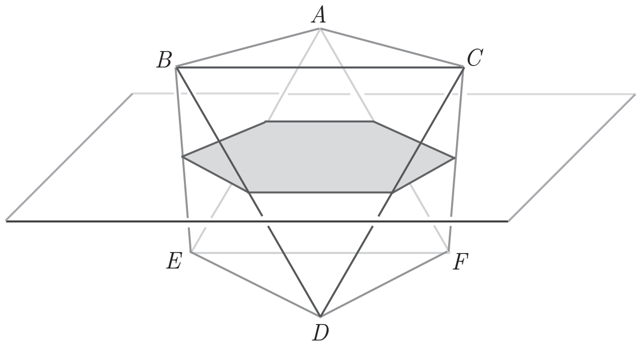



21. **Answer (C):** Because $x^{12}+ax^8+bx^4+c=p(x^4)$, the value of this polynomial is 0 if and only if

$$x^4=2009+9002\pi i \qquad \text{or} \qquad x^4=2009 \qquad \text{or} \qquad x^4=9002.$$

The first of these three equations has four distinct nonreal solutions, and the second and third each have two distinct nonreal solutions. Thus $p(x^4)=x^{12}+ax^8+bx^4+c$ has 8 distinct nonreal zeros.

22. **Answer (E):** Let $\triangle ABC$ and $\triangle DEF$ be the two faces of the octahedron parallel to the cutting plane. The plane passes through the midpoints of the six edges of the octahedron that are not sides of either of those triangles. Hence the intersection of the plane with the octahedron is an equilateral hexagon with side length $\frac{1}{2}$. Then by symmetry the hexagon is also equiangular and hence regular. The area of the hexagon is 6 times that of an equilateral triangle with side length $\frac{1}{2}$, so the area is $6\left(\frac{1}{2}\right)^2\frac{\sqrt{3}}{4}=\frac{3\sqrt{3}}{8}$. Therefore $a+b+c=3+3+8=14$.




23. **Answer (D):** Let $(h,k)$ be the vertex of the graph of $f$. Because the graph of $f$ intersects the $x$-axis twice, we can assume that $f(x)=a(x-h)^2+k$ with $\frac{-k}{a}>0$. Let $s=\sqrt{\frac{-k}{a}}$; then the $x$-intercepts of the graph of $f$ are $h\pm s$. Because $g(x)=-f(100-x)=-a(100-x-h)^2-k$, it follows that the $x$-intercepts of the graph of $g$ are $100-h\pm s$.

The graph of $g$ contains the point $(h,k)$; thus

$$k=f(h)=g(h)=-a(100-2h)^2-k,$$

from which $h=50\pm\frac{\sqrt{2}}{2}s$. Regardless of the sign in the expression for $h$, the four $x$-intercepts in order are

$$50-s\left(1+\frac{\sqrt{2}}{2}\right)<50-s\left(1-\frac{\sqrt{2}}{2}\right)<50+s\left(1-\frac{\sqrt{2}}{2}\right)<50+s\left(1+\frac{\sqrt{2}}{2}\right).$$

Because $x_3-x_2=150$, it follows that $150=s(2-\sqrt{2})$, that is $s=150\left(1+\frac{\sqrt{2}}{2}\right)$. Therefore $x_4-x_1=s(2+\sqrt{2})=450+300\sqrt{2}$, and then $m+n+p=450+300+2=752$.

**OR**

The graphs of $f$ and $g$ intersect the $x$-axis twice each. By symmetry, and because the graph of $g$ contains the vertex of $f$, we can assume $x_1$ and $x_3$ are the roots of $f$, and $x_2$ and $x_4$ are the roots of $g$. A point $(p, q)$ is on the graph of $f$ if and only if $(100-p, -q)$ is on the graph of $g$, so the two graphs are reflections of each other with respect to the point $(50, 0)$. Thus $x_2+x_3 = x_1+x_4 = 100$, and since $x_3 - x_2 = 150$, it follows that $x_2 = -25$ and $x_3 = 125$. The average of $x_1$ and $x_3 = 125$ is $h$. It follows that $x_1 = 2h-125$, from which $x_4 = 100-x_1 = 225-2h$, and $x_4 - x_1 = 350 - 4h$.

Moreover, $f(x) = a(x - x_1)(x - x_3) = a(x + 125 - 2h)(x - 125)$ and $g(x) = -f(100 - x) = -a(x + 25)(x + 2h - 225)$. The vertex of the graph of $f$ lies on the graph of $g$; thus

$$1 = \frac{f(h)}{g(h)} = \frac{(125 - h)(h - 125)}{-(h + 25)(3h - 225)},$$

from which $h = -25 \pm 75\sqrt{2}$. However, $h < x_2 < 0$; thus $h = -25 - 75\sqrt{2}$. Therefore $x_4 - x_1 = 450 + 300\sqrt{2}$ and then $m + n + p = 450 + 300 + 2 = 752$.

24. **Answer (E):** Define the *k-iterated logarithm* as follows: $\log_2^1 x = \log_2 x$ and $\log_2^{k+1} x = \log_2(\log_2^k x)$ for $k \geq 1$. Because $\log_2 T(n + 1) = T(n)$ for $n \geq 1$, it follows that $\log_2 A = T(2009)\log_2 T(2009) = T(2009)T(2008)$ and $\log_2 B = A \log_2 T(2009) = A \cdot T(2008)$. Then $\log_2^2 B = \log_2 A + \log_2 T(2008) = T(2009)T(2008) + T(2007)$. Now,

$$\log_2^3 B > \log_2(T(2009)T(2008)) > \log_2 T(2009) = T(2008),$$

and recursively for $k \geq 1$,

$$\log_2^{k+3} B > T(2008 - k).$$

In particular $\log_2^{2010} B > T(1) = 2$, and then $\log_2^{2012} B > 0$. Thus $\log_2^{2013} B$ is defined.

On the other hand, because $T(2007) < T(2008)T(2009)$ and $1 + T(2007) < T(2008)$, it follows that

$$\log_2^3 B < \log_2(2T(2008)T(2009)) = 1 + T(2007) + T(2008) < 2T(2008) \text{ and}$$
$$\log_2^4 B < \log_2(2T(2008)) = 1 + T(2007) < T(2008).$$

Applying $\log_2$ recursively for $k \geq 1$ we get

$$\log_2^{4+k} B < T(2008 - k).$$

In particular $\log_2^{2011} B < T(1) = 2$, and then $\log_2^{2013} B < 0$. Thus $\log_2^{2014} B$ is undefined.

25. **Answer (A):** Recognize the similarity between the recursion formula given and the trigonometric identity

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Also note that the first two terms of the sequence are tangents of familiar angles, namely $\frac{\pi}{4}$ and $\frac{\pi}{6}$. Let $c_1 = 3$, $c_2 = 2$, and $c_{n+2} = (c_n + c_{n+1}) \bmod 12$. We claim that the sequence $\{a_n\}$ satisfies $a_n = \tan\left(\frac{\pi c_n}{12}\right)$. Note that

$$\begin{aligned} a_1 &= 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi c_1}{12}\right) \text{ and} \\ a_2 &= \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi c_2}{12}\right). \end{aligned}$$

By induction on $n$, the formula for the tangent of the sum of two angles, and the fact that the period of $\tan x$ is $\pi$,

$$\begin{aligned} a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} &= \frac{\tan\left(\frac{\pi c_n}{12}\right) + \tan\left(\frac{\pi c_{n+1}}{12}\right)}{1 - \tan\left(\frac{\pi c_n}{12}\right)\tan\left(\frac{\pi c_{n+1}}{12}\right)} \\ &= \tan\left(\frac{\pi(c_n + c_{n+1})}{12}\right) = \tan\left(\frac{\pi c_{n+2}}{12}\right). \end{aligned}$$

The first few terms of the sequence $\{c_n\}$ are:

$$3, 2, 5, 7, 0, 7, 7, 2, 9, 11, 8, 7, 3, 10, 1, 11, 0, 11, 11, 10, 9, 7, 4, 11, 3, 2.$$

So the sequence $c_n$ is periodic with period 24. Because $2009 = 24 \cdot 83 + 17$, it follows that $c_{2009} = c_{17} = 0$. Thus $|a_{2009}| = |\tan\left(\frac{\pi c_{17}}{12}\right)| = 0$.

The problems and solutions in this contest were proposed by Bernardo Ábrego, Betsy Bennett, Thomas Butts, George Brauer, Steve Dunbar, Douglas Faires, Sister Josanne Furey, Gregory Galperin, John Haverhals, Elgin Johnston, Joe Kennedy, David Wells, LeRoy Wenstrom, and Woody Wenstrom.

*The*

## American Mathematics Competitions

*are Sponsored by*

### The Mathematical Association of America
### The Akamai Foundation

*Contributors*

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Math Zoom Academy
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.