THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

58th Annual American Mathematics Contest 12

AMC 12
CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 6, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (C):** Susan pays $(4)(0.75)(20) = 60$ dollars. Pam pays $(5)(0.70)(20) = 70$ dollars, so she pays $70 - 60 = 10$ more dollars than Susan.

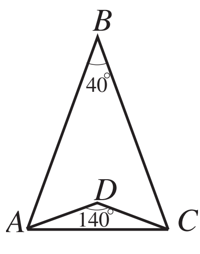
2. **Answer (D):** The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have
$$8000 = 4000h \quad \text{and} \quad h = 2 \text{ centimeters}$$

3. **Answer (A):** Let the smaller of the integers be x. Then the larger is $x + 2$. So $x + 2 = 3x$, from which $x = 1$. Thus the two integers are 1 and 3, and their sum is 4.

4. **Answer (A):** Kate rode for 30 minutes = 1/2 hour at 16 mph, so she rode 8 miles. She walked for 90 minutes = 3/2 hours at 4 mph, so she walked 6 miles. Therefore she covered a total of 14 miles in 2 hours, so her average speed was 7 mph.

5. **Answer (D):** After paying the federal taxes, Mr. Public had 80% of his inheritance money left. He paid 10% of that, or 8% of his inheritance, in state taxes. Hence his total tax bill was 28% of his inheritance, and his inheritance was \$10,500/0.28 = \$37,500.

6. **Answer (D):** Because $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = 70^\circ$.



Similarly,
$$\angle DAC = \frac{1}{2}(180^\circ - \angle ADC) = 20^\circ.$$
Thus $\angle BAD = \angle BAC - \angle DAC = 50^\circ$.

OR

Because $\triangle ABC$ and $\triangle ADC$ are isosceles triangles, applying the Exterior Angle Theorem to $\triangle ABD$ gives $\angle BAD = 70^\circ - 20^\circ = 50^\circ$.

7. **Answer (C):** Let D be the difference between consecutive terms of the sequence. Then $a = c - 2D$, $b = c - D$, $d = c + D$, and $e = c + 2D$, so

$$a + b + c + d + e = (c - 2D) + (c - D) + c + (c + D) + (c + 2D) = 5c.$$

Thus $5c = 30$, so $c = 6$.

To see that the values of the other terms cannot be found, note that the sequences $4, 5, 6, 7, 8$ and $10, 8, 6, 4, 2$ both satisfy the given conditions.

8. **Answer (C):** Consider the two chords with an endpoint at 5. The arc subtended by the angle determined by these chords extends from 10 to 12, so the degree measure of the arc is $(2/12)(360) = 60$. By the Central Angle Theorem, the degree measure of this angle is $(1/2)(60) = 30$. By symmetry, the degree measure of the angle at each vertex is 30.

9. **Answer (B):** Let w be Yan's walking speed, and let x and y be the distances from Yan to his home and to the stadium, respectively. The time required for Yan to walk to the stadium is y/w, and the time required for him to walk home is x/w. Because he rides his bicycle at a speed of $7w$, the time required for him to ride his bicycle from his home to the stadium is $(x + y)/(7w)$. Thus

$$\frac{y}{w} = \frac{x}{w} + \frac{x + y}{7w} = \frac{8x + y}{7w}.$$

As a consequence, $7y = 8x + y$, so $8x = 6y$. The required ratio is $x/y = 6/8 = 3/4$.

OR

Because we are interested only in the ratio of the distances, we may assume that the distance from Yan's home to the stadium is 1 mile. Let x be his present distance from his home. Imagine that Yan has a twin, Nay. While Yan walks to the stadium, Nay walks to their home and continues $1/7$ of a mile past their home. Because walking $1/7$ of a mile requires the same amount of time as riding 1 mile, Yan and Nay will complete their trips at the same time. Yan has walked $1 - x$ miles while Nay has walked $x + \frac{1}{7}$ miles, so $1 - x = x + \frac{1}{7}$. Thus $x = 3/7$, $1 - x = 4/7$, and the required ratio is $x/(1 - x) = 3/4$.

10. **Answer (A):** Let the sides of the triangle have lengths $3x$, $4x$, and $5x$. The triangle is a right triangle, so its hypotenuse is a diameter of the circle. Thus $5x = 2 \cdot 3 = 6$, so $x = 6/5$. The area of the triangle is

$$\frac{1}{2} \cdot 3x \cdot 4x = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

OR

A right triangle with side lengths 3, 4, and 5 has area $(1/2)(3)(4) = 6$. Because the given right triangle is inscribed in a circle with diameter 6, the hypotenuse of this triangle has length 6. Thus the sides of the given triangle are $6/5$ as long as those of a $3-4-5$ triangle, and its area is $(6/5)^2$ times that of a $3-4-5$ triangle. The area of the given triangle is

$$\left(\frac{6}{5}\right)^2(6) = \frac{216}{25} = 8.64.$$

11. **Answer (D):** A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S = 111k = 3 \cdot 37k$, so S must be divisible by 37. To see that S need not be divisible by any larger prime, note that the sequence 123, 231, 312 gives $S = 666 = 2 \cdot 3^2 \cdot 37$.

12. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2) \cdot (1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{5}{8}.$$

13. **Answer (B):** The point (a, b) is the foot of the perpendicular from $(12, 10)$ to the line $y = -5x + 18$. The perpendicular has slope $\frac{1}{5}$, so its equation is

$$y = 10 + \frac{1}{5}(x - 12) = \frac{1}{5}x + \frac{38}{5}.$$

The x-coordinate at the foot of the perpendicular satisfies the equation

$$\frac{1}{5}x + \frac{38}{5} = -5x + 18,$$

so $x = 2$ and $y = -5 \cdot 2 + 18 = 8$. Thus $(a, b) = (2, 8)$, and $a + b = 10$.

OR

If the mouse is at $(x, y) = (x, 18 - 5x)$, then the square of the distance from the mouse to the cheese is

$$(x - 12)^2 + (8 - 5x)^2 = 26\left(x^2 - 4x + 8\right) = 26\left((x - 2)^2 + 4\right).$$

The value of this expression is smallest when $x = 2$, so the mouse is closest to the cheese at the point $(2, 8)$, and $a + b = 2 + 8 = 10$.

14. **Answer (C):** If 45 is expressed as a product of five distinct integer factors, the absolute value of the product of any four is at least $|(-3)(-1)(1)(3)| = 9$, so no factor can have an absolute value greater than 5. Thus the factors of the given expression are five of the integers $\pm1, \pm3$, and ±5. The product of all six of these is $-225 = (-5)(45)$, so the factors are -3, -1, 1, 3, and 5. The corresponding values of a, b, c, d, and e are 9, 7, 5, 3, and 1, and their sum is 25.

15. **Answer (E):** The mean of the augmented set is $(28 + n)/5$. If $n < 6$, the median of that set is 6, so $28 + n = 5 \cdot 6$, and $n = 2$. If $6 < n < 9$, the median is n, so $28 + n = 5n$, and $n = 7$. If $n > 9$, the median is 9, so $28 + n = 5 \cdot 9$, and $n = 17$. Thus the sum of all possible values of n is $2 + 7 + 17 = 26$.

16. **Answer (C):** The set of the three digits of such a number can be arranged to form an increasing arithmetic sequence. There are 8 possible sequences with a common difference of 1, since the first term can be any of the digits 0 through 7. There are 6 possible sequences with a common difference of 2, 4 with a common difference of 3, and 2 with a common difference of 4. Hence there are 20 possible arithmetic sequences. Each of the 4 sets that contain 0 can be arranged to form $2 \cdot 2! = 4$ different numbers, and the 16 sets that do not contain 0 can be arranged to form $3! = 6$ different numbers. Thus there are a total of $4 \cdot 4 + 16 \cdot 6 = 112$ numbers with the required properties.

17. **Answer (B):** Square both sides of both given equations to obtain

$$\sin^2 a + 2\sin a \sin b + \sin^2 b = 5/3 \quad \text{and} \quad \cos^2 a + 2\cos a \cos b + \cos^2 b = 1.$$

Then add corresponding sides of the resulting equations to obtain

$$\left(\sin^2 a + \cos^2 a\right) + \left(\sin^2 b + \cos^2 b\right) + 2(\sin a \sin b + \cos a \cos b) = \frac{8}{3}.$$

Because $\sin^2 a + \cos^2 a = \sin^2 b + \cos^2 b = 1$, it follows that

$$\cos(a - b) = \sin a \sin b + \cos a \cos b = \frac{1}{3}.$$

One ordered pair (a, b) that satisfies the given condition is approximately $(0.296, 1.527)$.

18. **Answer (D):** Because $f(x)$ has real coefficients and $2i$ and $2 + i$ are zeros, so are their conjugates $-2i$ and $2 - i$. Therefore

$$\begin{aligned} f(x) = (x + 2i)(x - 2i)(x - (2 + i))(x - (2 - i)) &= (x^2 + 4)(x^2 - 4x + 5) \\ &= x^4 - 4x^3 + 9x^2 - 16x + 20. \end{aligned}$$

Hence $a + b + c + d = -4 + 9 - 16 + 20 = 9$.

OR

As in the first solution,

$$f(x) = (x + 2i)(x - 2i)(x - (2 + i))(x - (2 - i)),$$

so

$$a+b+c+d = f(1)-1 = (1+2i)(1-2i)(-1-i)(-1+i)-1 = (1+4)(1+1)-1 = 9.$$

19. **Answer (E):** Let h be the length of the altitude from A in $\triangle ABC$. Then

$$2007 = \frac{1}{2} \cdot BC \cdot h = \frac{1}{2} \cdot 223 \cdot h,$$

so $h = 18$. Thus A is on one of the lines $y = 18$ or $y = -18$. Line DE has equation $x - y - 300 = 0$. Let A have coordinates (a, b). By the formula for the distance from a point to a line, the distance from A to line DE is $|a-b-300|/\sqrt{2}$. The area of $\triangle ADE$ is

$$7002 = \frac{1}{2} \cdot \frac{|a - b - 300|}{\sqrt{2}} \cdot DE = \frac{1}{2} \cdot \frac{|a \pm 18 - 300|}{\sqrt{2}} \cdot 9\sqrt{2}.$$

Thus $a = \pm 18 \pm 1556 + 300$, and the sum of the four possible values of a is $4 \cdot 300 = 1200$.

OR

As above, conclude that A is on one of the lines $y = \pm 18$. By similar reasoning, A is on one of two particular lines l_1 and l_2 parallel to $\overline{DE}$. Therefore there are four possible positions for A, determined by the intersections of the lines $y = 18$ and $y = -18$ with each of l_1 and l_2. Let the line $y = 18$ intersect l_1 and l_2 in points (x_1, y_1) and (x_2, y_2), and let the line $y = -18$ intersect l_1 and l_2 in points (x_3, y_3) and (x_4, y_4). The four points of intersection are the vertices of a parallelogram, and the center of the parallelogram has x-coordinate $(1/4)(x_1 + x_2 + x_3 + x_4)$. The center is the intersection of the line $y = 0$ and line DE. Because line DE has equation $y = x - 300$, the center of the parallelogram is $(300, 0)$. Thus the sum of all possible x-coordinates of A is $4 \cdot 300 = 1200$.

20. **Answer (B):** Removing the corners removes two segments of equal length from each edge of the cube. Call that length x. Then each octagon has side length $\sqrt{2}x$, and the cube has edge length $1 = \left(2 + \sqrt{2}\right) x$, so

$$x = \frac{1}{2 + \sqrt{2}} = \frac{2 - \sqrt{2}}{2}.$$

Each removed corner is a tetrahedron whose altitude is x and whose base is an isosceles right triangle with leg length x. Thus the total volume of the eight tetrahedra is

$$8 \cdot \frac{1}{3} \cdot x \cdot \frac{1}{2}x^2 = \frac{1}{6}\left(2 - \sqrt{2}\right)^3 = \frac{10 - 7\sqrt{2}}{3}.$$

21. **Answer (A):** The product of the zeros of f is c/a, and the sum of the zeros is $-b/a$. Because these two numbers are equal, $c = -b$, and the sum of the coefficients is $a + b + c = a$, which is the coefficient of x^2. To see that none of the other choices is correct, let $f(x) = -2x^2 - 4x + 4$. The zeros of f are $-1 \pm \sqrt{3}$, so the sum of the zeros, the product of the zeros, and the sum of the coefficients are all -2. However, the coefficient of x is -4, the y-intercept is 4, the x-intercepts are $-1 \pm \sqrt{3}$, and the mean of the x-intercepts is -1.

22. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy

$$n \ge 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n),$ and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n),$ and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

23. **Answer (A):** Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Then $AB = 6 = |p - q|$. Because $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$. Thus $|q^2 - q| = 6$, and the only positive solution is $q = 3$. Note that $C = (q, 3\log_a q)$, so $BC = 6 = \log_a q$, from which $a^6 = q = 3$ and $a = \sqrt[6]{3}$.

24. **Answer (D):** Note that $F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$. For each n, $\sin nx \ge 0$ on each interval $[(2k-2)\pi/n, (2k-1)\pi/n]$ where k is a positive integer and $2k - 1 \le n$. The number of such intervals is $n/2$ if n is even and $(n+1)/2$ if n is odd. The graphs intersect twice on each interval unless $\sin x = 1 = \sin nx$ at some point in the interval, in which case the graphs intersect once. This last equation is satisfied if and only if $n \equiv 1 \pmod 4$ and the interval contains $\pi/2$. If n is even, this count does not include the point of intersection at $(\pi, 0)$. Therefore $F(n) = 2(n/2) + 1 = n + 1$ if n is even, $F(n) = 2(n+1)/2 = n + 1$ if $n \equiv 3 \pmod 4$, and $F(n) = n$ if $n \equiv 1 \pmod 4$. Hence

$$\sum_{n=2}^{2007} F(n) = \left(\sum_{n=2}^{2007}(n+1)\right) - \left\lfloor \frac{2007-1}{4} \right\rfloor = \frac{(2006)(3+2008)}{2} - 501 = 2{,}016{,}532.$$

25. **Answer (E):** For each positive integer n, let $S_n = \{k : 1 \leq k \leq n\}$, and let c_n be the number of spacy subsets of S_n. Then $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. For $n \geq 4$, the spacy subsets of S_n can be partitioned into two types: those that contain n and those that do not. Those that do not contain n are precisely the spacy subsets of S_{n-1}. Those that contain n do not contain either $n-1$ or $n-2$ and are therefore in one-to-one correspondence with the spacy subsets of S_{n-3}. It follows that $c_n = c_{n-3} + c_{n-1}$. Thus the first twelve terms in the sequence (c_n) are 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, and there are $c_{12} = 129$ spacy subsets of S_{12}.

OR

Note that each spacy subset of S_{12} contains at most 4 elements. For each such subset $a_1, a_2, \ldots, a_k$, let $b_1 = a_1 - 1$, $b_j = a_j - a_{j-1} - 3$ for $2 \leq j \leq k$, and $b_{k+1} = 12 - a_k$. Then $b_j \geq 0$ for $1 \leq j \leq k+1$, and

$$b_1 + b_2 + \cdots + b_{k+1} = 12 - 1 - 3(k-1) = 14 - 3k.$$

The number of solutions for $(b_1, b_2, \ldots, b_{k+1})$ is $\binom{14-2k}{k}$ for $0 \leq k \leq 4$, so the number of spacy subsets of S_{12} is

$$\binom{14}{0} + \binom{12}{1} + \binom{10}{2} + \binom{8}{3} + \binom{6}{4} = 1 + 12 + 45 + 56 + 15 = 129.$$

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