The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

57th Annual American Mathematics Contest 12

AMC 12 – Contest A

Solutions Pamphlet

Tuesday, JANUARY 31, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **(A)** Five sandwiches cost $5 \cdot 3 = 15$ dollars and eight sodas cost $8 \cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

2. **(C)** By the definition we have

$$h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

3. **(B)** Mary is $(3/5)(30) = 18$ years old.

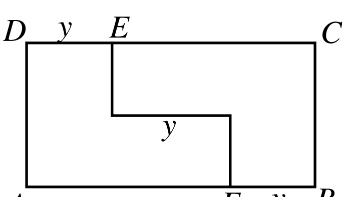
4. **(E)** The largest possible sum of the two digits representing the minutes is $5+9 = 14$, occurring at 59 minutes past each hour. The largest possible single digit that can represent the hour is 9. This exceeds the largest possible sum of two digits that can represent the hour, which is $1 + 2 = 3$. Therefore, the largest possible sum of all the digits is $14 + 9 = 23$, occurring at 9:59.

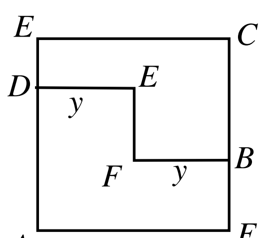
5. **(D)** Each slice of plain pizza cost \$1. Dave paid \$2 for the anchovies in addition to \$5 for the five slices of pizza that he ate, for a total of \$7. Doug paid only \$3 for the three slices of pizza that he ate. Hence Dave paid $7 - 3 = 4$ dollars more than Doug.

6. **(A)** Let E represent the end of the cut on $\overline{DC}$, and let F represent the end of the cut on $\overline{AB}$. For a square to be formed, we must have

$$DE = y = FB \quad \text{and} \quad DE + y + FB = 18, \quad \text{so} \quad y = 6.$$

The rectangle that is formed by this cut is indeed a square, since the original rectangle has area $8 \cdot 18 = 144$, and the rectangle that is formed by this cut has a side of length $2 \cdot 6 = 12 = \sqrt{144}$.





7. **(B)** Let Danielle be x years old. Sally is 40% younger, so she is $0.6x$ years old. Mary is 20% older than Sally, so Mary is $1.2(0.6x) = 0.72x$ years old. The sum of their ages is $23.2 = x + 0.6x + 0.72x = 2.32x$ years, so $x = 10$. Therefore Mary's age is $0.72x = 7.2$ years, and she will be 8 on her next birthday.

8. **(C)** First note that, in general, the sum of n consecutive integers is n times their median. If the sum is 15, we have the following cases:

   if $n = 2$, then the median is 7.5 and the two integers are 7 and 8;

   if $n = 3$, then the median is 5 and the three integers are 4, 5, and 6;

   if $n = 5$, then the median is 3 and the five integers are 1, 2, 3, 4, and 5.

   Because the sum of four consecutive integers is even, 15 cannot be written in such a manner. Also, the sum of more than five consecutive integers must be more than $1+2+3+4+5 = 15$. Hence there are 3 sets satisfying the condition.

   Note: It can be shown that the number of sets of two or more consecutive positive integers having a sum of k is equal to the number of odd positive divisors of k, excluding 1.

9. **(A)** Let p be the cost (in cents) of a pencil, and let s be the cost (in cents) of a set of one pencil and one eraser. Because Oscar buys 3 sets and 10 extra pencils for \$1.00, we have

   $$3s + 10p = 100.$$

   Thus $3s$ is a multiple of 10 that is less than 100, so s is 10, 20, or 30. The corresponding values of p are 7, 4, and 1. Since the cost of a pencil is more than half the cost of the set, the only possibility is $s = 10$.

10. **(E)** Suppose that $k = \sqrt{120 - \sqrt{x}}$ is an integer. Then $0 \le k \le \sqrt{120}$, and because k is an integer, we have $0 \le k \le 10$. Thus there are 11 possible integer values of k. For each such k, the corresponding value of x is $\left(120 - k^2\right)^2$. Because $\left(120 - k^2\right)^2$ is positive and decreasing for $0 \le k \le 10$, the 11 values of x are distinct.

11. **(C)** The equation $(x+y)^2 = x^2 + y^2$ is equivalent to $x^2 + 2xy + y^2 = x^2 + y^2$, which reduces to $xy = 0$. Thus the graph of the equation consists of the two lines that are the coordinate axes.

12. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are $16\,\text{cm}, 15\,\text{cm}, \dots, 1\,\text{cm}$. Thus the total distance is

    $$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

    OR

    The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

    $$(3+4+5+\cdots+20) - 2\cdot 17 = (1+2+3+4+5+\cdots+20) - 3 - 34 = \frac{20 \cdot 21}{2} - 37 = 173\,\text{cm}.$$

13. **(E)** Let r, s, and t be the radii of the circles centered at A, B, and C, respectively. Then $r+s=3$, $r+t=4$, and $s+t=5$, from which $r=1$, $s=2$, and $t=3$. Thus the sum of the areas of the circles is

$$\pi(1^2+2^2+3^2)=14\pi.$$

14. **(C)** If a debt of D dollars can be resolved in this way, then integers p and g must exist with

$$D=300p+210g=30(10p+7g).$$

As a consequence, D must be a multiple of 30, so no positive debt of less than \$30 can be resolved. A debt of \$30 can be resolved since

$$30=300(-2)+210(3).$$

This is done by giving 3 goats and receiving 2 pigs.

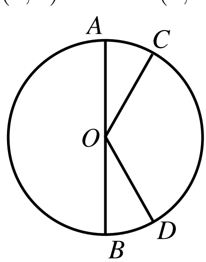
15. **(A)** Because $\cos x=0$ and $\cos(x+z)=1/2$, it follows that $x=m\pi/2$ for some odd integer m and $x+z=2n\pi\pm\pi/3$ for some integer n. Therefore

$$z=2n\pi-\frac{m\pi}{2}\pm\frac{\pi}{3}=k\pi+\frac{\pi}{2}\pm\frac{\pi}{3}$$

for some integer k. The smallest value of k that yields a positive value for z is 0, and the smallest positive value of z is $\pi/2-\pi/3=\pi/6$.

OR

Let O denote the center of the unit circle. Because $\cos x=0$, the terminal side of an angle of measure x, measured counterclockwise from the positive x-axis, intersects the circle at $A=(0,1)$ or $B=(0,-1)$.



Because $\cos(x+z)=1/2$, the terminal side of an angle of measure $x+z$ intersects the circle at $C=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ or $D=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$. Thus all angles of positive measure $z=(x+z)-x$ can be measured counterclockwise from either $\overline{OA}$ or $\overline{OB}$ to either $\overline{OC}$ or $\overline{OD}$. The smallest such angle is $\angle BOD$, which has measure $\pi/6$ and is attained, for example, when $x=-\pi/2$ and $x+z=-\pi/3$.

16. **(B)** Radii $\overline{AC}$ and $\overline{BD}$ are each perpendicular to $\overline{CD}$. By the Pythagorean Theorem,

$$CE = \sqrt{5^2 - 3^2} = 4.$$

Because $\triangle ACE$ and $\triangle BDE$ are similar,

$$\frac{DE}{CE} = \frac{BD}{AC}, \quad \text{so} \quad DE = CE \cdot \frac{BD}{AC} = 4 \cdot \frac{8}{3} = \frac{32}{3}.$$

Therefore

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

17. **(B)** Let $B = (0,0)$, $C = (s,0)$, $A = (0,s)$, $D = (s,s)$, and $E = \left(s + \frac{r}{\sqrt{2}}, s + \frac{r}{\sqrt{2}}\right)$. Apply the Pythagorean Theorem to $\triangle AFE$ to obtain

$$r^2 + \left(9 + 5\sqrt{2}\right) = \left(s + \frac{r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2,$$

from which $9 + 5\sqrt{2} = s^2 + rs\sqrt{2}$. Because r and s are rational, it follows that $s^2 = 9$ and $rs = 5$, so $r/s = 5/9$.

OR

Extend $\overline{AD}$ past D to meet the circle at $G \neq D$. Because E is collinear with B and D, $\triangle EDG$ is an isosceles right triangle. Thus $DG = r\sqrt{2}$. By the Power of a Point Theorem,

$$9 + 5\sqrt{2} = AF^2 = AD \cdot AG = AD \cdot (AD + DG) = s\left(s + r\sqrt{2}\right) = s^2 + rs\sqrt{2}.$$

As in the first solution, conclude that $r/s = 5/9$.

18. **(E)** The conditions on f imply that both

$$x = f(x) + f\left(\frac{1}{x}\right) \quad \text{and} \quad \frac{1}{x} = f\left(\frac{1}{x}\right) + f\left(\frac{1}{1/x}\right) = f\left(\frac{1}{x}\right) + f(x).$$

Thus if x is in the domain of f, then $x = 1/x$, so $x = \pm 1$.

The conditions are satisfied if and only if $f(1) = 1/2$ and $f(-1) = -1/2$.

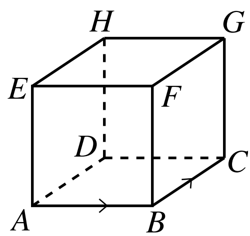
19. **(E)** The slope of the line l containing the centers of the circles is $5/12 = \tan\theta$, where θ is the acute angle between the x-axis and line l. The equation of line l is $y - 4 = (5/12)(x - 2)$. This line and the two common external tangents are concurrent. Because one of these tangents is the x-axis, the point of intersection is the x-intercept of line l, which is $(-38/5, 0)$. The acute angle between the x-axis and the other tangent is 2θ, so the slope of that tangent is

$$\tan 2\theta = 2 \cdot \frac{5/12}{1 - (5/12)^2} = \frac{120}{119}.$$

Thus the equation of that tangent is $y = (120/119)\,(x + (38/5))$, and

$$b = \frac{120}{119} \cdot \frac{38}{5} = \frac{912}{119}.$$

20. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as $A, B,$ and C, in that order. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, and D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug can travel from C to D and then from D to H. Each of these occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow E \to F \to G \\ \searrow G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3} + \frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$.

21. **(E)** For $j = 1$ and 2, the given inequality is equivalent to

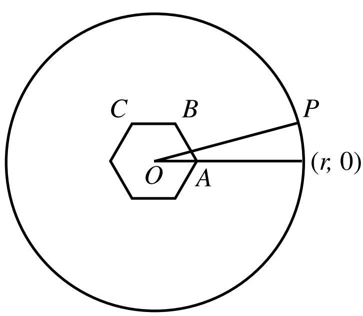
$$j + x^2 + y^2 \leq 10^j (x + y),$$

or to

$$\left(x - \frac{10^j}{2}\right)^2 + \left(y - \frac{10^j}{2}\right)^2 \leq \frac{10^{2j}}{2} - j,$$

provided that $x + y > 0$. These inequalities define regions bounded by circles. For $j = 1$ the circle has center $(5, 5)$ and radius 7. For $j = 2$ the circle has center $(50, 50)$ and radius $\sqrt{4998}$. In each case the center is on the line $y = x$ in the first quadrant, and the radius is less than the distance from the center to the origin. Thus $x + y > 0$ at each interior point of each circle, as was required to ensure the equivalence of the inequalities. The squares of the radii of the circles are 49 and 4998 for $j = 1$ and 2, respectively. Therefore the ratio of the area of S_2 to that of S_1 is $(4998\pi)/(49\pi) = 102$.

22. **(D)** Place the hexagon in a coordinate plane with center at the origin O and vertex A at $(2, 0)$. Let B, C, D, E, and F be the other vertices in counterclockwise order.



Corresponding to each vertex of the hexagon, there is an arc on the circle from which only the two sides meeting at that vertex are visible. The given probability condition implies that those arcs have a combined degree measure of $180°$, so by symmetry each is $30°$. One such arc is centered at $(r, 0)$. Let P be the endpoint of this arc in the upper half-plane. Then $\angle POA = 15°$. Side $\overline{BC}$ is visible from points immediately above P, so P is collinear with B and C. Because the perpendicular distance from O to $\overline{BC}$ is $\sqrt{3}$, we have

$$\sqrt{3} = r \sin 15° = r \sin(45° - 30°) = r \left(\sin 45° \cos 30° - \sin 30° \cos 45°\right).$$

So

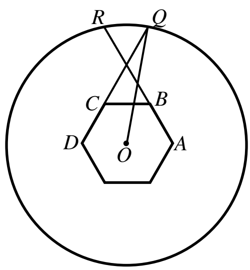
$$\sqrt{3} = r \cdot \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = r \cdot \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Therefore

$$r = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \cdot \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \sqrt{18}+\sqrt{6} = 3\sqrt{2}+\sqrt{6}.$$

OR

Call the hexagon $ABCDEF$. Side $\overline{AB}$ is visible from point X if and only if X lies in the half-plane that is in the exterior of the hexagon and that is determined by the line AB. The region from which the three sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are visible is the intersection of three such half-planes.



Let rays AB and DC intersect the circle at R and Q, respectively. Then QR is one of the six arcs of the circle from which three sides are visible. Symmetry implies that the six arcs are congruent, and because the given probability is $1/2$, the measure of each arc is $30°$. Let O be the center of the hexagon and the circle. Then $\angle QOR = 30°$, so

$$\angle QOC = \angle QOR + \frac{1}{2}(\angle BOC - \angle QOR) = 30° + \frac{1}{2}(60° - 30°) = 45°.$$

Thus

$$\angle OQC = 180° - \angle QOC - \angle OCQ = 15°.$$

Apply the Law of Sines in $\triangle OQC$ to obtain

$$\frac{r}{\sin 120°} = \frac{2}{\sin 15°}, \quad \text{and then} \quad r = \frac{\sqrt{3}}{\sin 15°}.$$

Then proceed as in the first solution.

23. **(B)** For every sequence $S = (a_1, a_2, \ldots, a_n)$ of at least three terms,

$$A^2(S) = \left(\frac{a_1+2a_2+a_3}{4}, \frac{a_2+2a_3+a_4}{4}, \ldots, \frac{a_{n-2}+2a_{n-1}+a_n}{4}\right).$$

Thus for $m = 1$ and 2, the coefficients of the terms in the numerator of $A^m(S)$ are the binomial coefficients $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$, and the denominator is 2^m. Because $\binom{m}{r} + \binom{m}{r+1} = \binom{m+1}{r+1}$ for all integers $r \ge 0$, the coefficients of the terms in the numerators of $A^{m+1}(S)$ are $\binom{m+1}{0}, \binom{m+1}{1}, \ldots, \binom{m+1}{m+1}$ for $2 \le m \le n-2$. The definition implies that the denominator of each term in $A^{m+1}(S)$ is 2^{m+1}. For the given sequence, the sole term in $A^{100}(S)$ is

$$\frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} a_{m+1} = \frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} x^m = \frac{1}{2^{100}} (x+1)^{100}.$$

Therefore

$$\left(\frac{1}{2^{50}}\right) = A^{100}(S) = \left(\frac{(1+x)^{100}}{2^{100}}\right),$$

so $(1+x)^{100} = 2^{50}$, and because $x > 0$, we have $x = \sqrt{2} - 1$.

24. **(D)** There is exactly one term in the simplified expression for every monomial of the form $x^a y^b z^c$, where a, b, and c are non-negative integers, a is even, and $a + b + c = 2006$. There are 1004 even values of a with $0 \le a \le 2006$. For each such value, b can assume any of the $2007 - a$ integer values between 0 and $2006 - a$, inclusive, and the value of c is then uniquely determined as $2006 - a - b$. Thus the number of terms in the simplified expression is

$$(2007 - 0) + (2007 - 2) + \cdots + (2007 - 2006) = 2007 + 2005 + \cdots + 1.$$

This is the sum of the first 1004 odd positive integers, which is $1004^2 = 1{,}008{,}016$.

OR

The given expression is equal to

$$\sum \frac{2006!}{a!b!c!} \left(x^a y^b z^c + x^a (-y)^b (-z)^c\right),$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c = 2006$. Because the number of non-negative integer solutions of $a + b + c = k$ is $\binom{k+2}{2}$, the sum is taken over $\binom{2008}{2}$ terms, but those for which b and c have opposite parity have a sum of zero. If b is odd and c is even, then a is odd, so $a = 2A+1, b = 2B+1,$ and $c = 2C$ for some non-negative integers $A, B,$ and C. Therefore $2A + 1 + 2B + 1 + 2C = 2006$, so $A + B + C = 1002$. Because the last equation has $\binom{1004}{2}$ non-negative integer solutions, there are $\binom{1004}{2}$ terms for which b is odd and c is even. The number of terms for which b is even and c is odd is the same. Thus the number of terms in the simplified expression is

$$\binom{2008}{2} - 2\binom{1004}{2} = 1004 \cdot 2007 - 1004 \cdot 1003 = 1004^2 = 1{,}008{,}016.$$

25. **(E)** For $1 \le k \le 15$, the k-element sets with properties (1) and (2) are the k-element subsets of $U_k = \{k, k+1, \ldots, 15\}$ that contain no two consecutive integers. If $\{a_1, a_2, \ldots, a_k\}$ is such a set, with its elements listed in increasing order, then $\{a_1 + k - 1, a_2 + k - 2, \ldots, a_{k-1} + 1, a_k\}$ is a k-element subset of U_{2k-1}. Conversely, if $\{b_1, b_2, \ldots, b_k\}$ is a k-element subset of U_{2k-1}, with its elements listed in increasing order, then $\{b_1 - k + 1, b_2 - k + 2, \ldots, b_{k-1} - 1, b_k\}$ is a set with properties (1) and (2). Thus for each k, the number of k-element sets with properties (1) and (2) is equal to the number of k-element subsets of the $(17 - 2k)$-element set U_{2k-1}. Because $k \le 17 - 2k$ only if $k \le 5$, the total number of such sets is

$$\sum_{k=1}^{5} \binom{17-2k}{k} = \binom{15}{1} + \binom{13}{2} + \binom{11}{3} + \binom{9}{4} + \binom{7}{5} = 15 + 78 + 165 + 126 + 21 = 405.$$

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