

**Tuesday, JANUARY 31, 2006**

57<sup>th</sup> Annual American Mathematics Contest 12

**AMC 12**



**Contest A**

**The MATHEMATICAL ASSOCIATION OF AMERICA  
American Mathematics Competitions**

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have 75 MINUTES to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

*Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 24<sup>th</sup> annual American Invitational Mathematics Examination (AIME) on Tuesday, March 7, 2006 or Wednesday, March 22, 2006. More details about the AIME and other information are on the back page of this test booklet.*

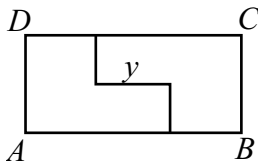
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The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.

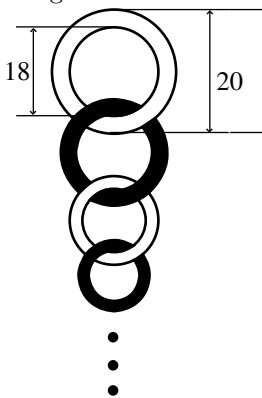
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1. Sandwiches at Joe's Fast Food cost \$3 each and sodas cost \$2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?
- (A) 31      (B) 32      (C) 33      (D) 34      (E) 35
2. Define  $x \otimes y = x^3 - y$ . What is  $h \otimes (h \otimes h)$ ?
- (A)  $-h$       (B) 0      (C)  $h$       (D)  $2h$       (E)  $h^3$
3. The ratio of Mary's age to Alice's age is 3 : 5. Alice is 30 years old. How old is Mary?
- (A) 15      (B) 18      (C) 20      (D) 24      (E) 50
4. A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?
- (A) 17      (B) 19      (C) 21      (D) 22      (E) 23
5. Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half of the pizza. The cost of a plain pizza was \$8, and there was an additional cost of \$2 for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each then paid for what he had eaten. How many more dollars did Dave pay than Doug?
- (A) 1      (B) 2      (C) 3      (D) 4      (E) 5
6. The  $8 \times 18$  rectangle  $ABCD$  is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is  $y$ ?



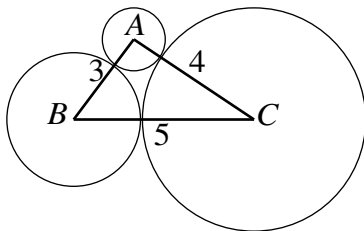
- (A) 6      (B) 7      (C) 8      (D) 9      (E) 10
7. Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?
- (A) 7      (B) 8      (C) 9      (D) 10      (E) 11
8. How many sets of two or more consecutive positive integers have a sum of 15?
- (A) 1      (B) 2      (C) 3      (D) 4      (E) 5

9. Oscar buys 13 pencils and 3 erasers for \$1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?
- (A) 10    (B) 12    (C) 15    (D) 18    (E) 20
10. For how many real values of  $x$  is  $\sqrt{120 - \sqrt{x}}$  an integer?
- (A) 3    (B) 6    (C) 9    (D) 10    (E) 11
11. Which of the following describes the graph of the equation  $(x + y)^2 = x^2 + y^2$ ?
- (A) the empty set    (B) one point    (C) two lines    (D) a circle  
(E) the entire plane
12. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

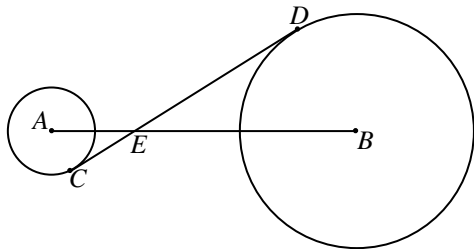


- (A) 171    (B) 173    (C) 182    (D) 188    (E) 210

13. The vertices of a 3–4–5 right triangle are the centers of three mutually externally tangent circles, as shown. What is the sum of the areas of these circles?

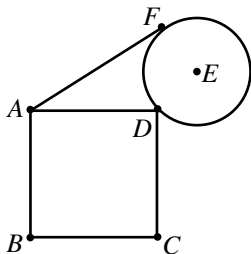


- (A)  $12\pi$     (B)  $\frac{25\pi}{2}$     (C)  $13\pi$     (D)  $\frac{27\pi}{2}$     (E)  $14\pi$
14. Two farmers agree that pigs are worth \$300 and that goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with “change” received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?
- (A) \$5    (B) \$10    (C) \$30    (D) \$90    (E) \$210
15. Suppose  $\cos x = 0$  and  $\cos(x + z) = 1/2$ . What is the smallest possible positive value of  $z$ ?
- (A)  $\frac{\pi}{6}$     (B)  $\frac{\pi}{3}$     (C)  $\frac{\pi}{2}$     (D)  $\frac{5\pi}{6}$     (E)  $\frac{7\pi}{6}$
16. Circles with centers  $A$  and  $B$  have radii 3 and 8, respectively. A common internal tangent intersects the circles at  $C$  and  $D$ , respectively. Lines  $AB$  and  $CD$  intersect at  $E$ , and  $AE = 5$ . What is  $CD$ ?



- (A) 13    (B)  $\frac{44}{3}$     (C)  $\sqrt{221}$     (D)  $\sqrt{255}$     (E)  $\frac{55}{3}$

17. Square  $ABCD$  has side length  $s$ , a circle centered at  $E$  has radius  $r$ , and  $r$  and  $s$  are both rational. The circle passes through  $D$ , and  $D$  lies on  $\overline{BE}$ . Point  $F$  lies on the circle, on the same side of  $\overline{BE}$  as  $A$ . Segment  $AF$  is tangent to the circle, and  $AF = \sqrt{9 + 5\sqrt{2}}$ . What is  $r/s$ ?

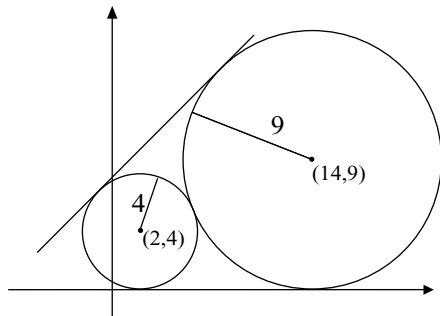


- (A)  $\frac{1}{2}$     (B)  $\frac{5}{9}$     (C)  $\frac{3}{5}$     (D)  $\frac{5}{3}$     (E)  $\frac{9}{5}$
18. The function  $f$  has the property that for each real number  $x$  in its domain,  $1/x$  is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x.$$

What is the largest set of real numbers that can be in the domain of  $f$ ?

- (A)  $\{x \mid x \neq 0\}$     (B)  $\{x \mid x < 0\}$     (C)  $\{x \mid x > 0\}$   
 (D)  $\{x \mid x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$     (E)  $\{-1, 1\}$
19. Circles with centers  $(2, 4)$  and  $(14, 9)$  have radii 4 and 9, respectively. The equation of a common external tangent to the circles can be written in the form  $y = mx + b$  with  $m > 0$ . What is  $b$ ?



- (A)  $\frac{908}{119}$     (B)  $\frac{909}{119}$     (C)  $\frac{130}{17}$     (D)  $\frac{911}{119}$     (E)  $\frac{912}{119}$

20. A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

(A)  $\frac{1}{2187}$     (B)  $\frac{1}{729}$     (C)  $\frac{2}{243}$     (D)  $\frac{1}{81}$     (E)  $\frac{5}{243}$

21. Let

$$S_1 = \{(x, y) \mid \log_{10}(1 + x^2 + y^2) \leq 1 + \log_{10}(x + y)\}$$

and

$$S_2 = \{(x, y) \mid \log_{10}(2 + x^2 + y^2) \leq 2 + \log_{10}(x + y)\}.$$

What is the ratio of the area of  $S_2$  to the area of  $S_1$ ?

(A) 98    (B) 99    (C) 100    (D) 101    (E) 102

22. A circle of radius  $r$  is concentric with and outside a regular hexagon of side length 2. The probability that three entire sides of the hexagon are visible from a randomly chosen point on the circle is  $1/2$ . What is  $r$ ?

(A)  $2\sqrt{2} + 2\sqrt{3}$     (B)  $3\sqrt{3} + \sqrt{2}$     (C)  $2\sqrt{6} + \sqrt{3}$     (D)  $3\sqrt{2} + \sqrt{6}$   
 (E)  $6\sqrt{2} - \sqrt{3}$

23. Given a finite sequence  $S = (a_1, a_2, \dots, a_n)$  of  $n$  real numbers, let  $A(S)$  be the sequence

$$\left( \frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \dots, \frac{a_{n-1} + a_n}{2} \right)$$

of  $n-1$  real numbers. Define  $A^1(S) = A(S)$  and, for each integer  $m$ ,  $2 \leq m \leq n-1$ , define  $A^m(S) = A(A^{m-1}(S))$ . Suppose  $x > 0$ , and let  $S = (1, x, x^2, \dots, x^{100})$ . If  $A^{100}(S) = (1/2^{50})$ , then what is  $x$ ?

(A)  $1 - \frac{\sqrt{2}}{2}$     (B)  $\sqrt{2} - 1$     (C)  $\frac{1}{2}$     (D)  $2 - \sqrt{2}$     (E)  $\frac{\sqrt{2}}{2}$

24. The expression

$$(x + y + z)^{2006} + (x - y - z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

(A) 6018    (B) 671,676    (C) 1,007,514    (D) 1,008,016    (E) 2,015,028

25. How many non-empty subsets  $S$  of  $\{1, 2, 3, \dots, 15\}$  have the following two properties?

- (1) No two consecutive integers belong to  $S$ .
- (2) If  $S$  contains  $k$  elements, then  $S$  contains no number less than  $k$ .

(A) 277      (B) 311      (C) 376      (D) 377      (E) 405

## WRITE TO US!

*Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:*

American Mathematics Competitions  
University of Nebraska, P.O. Box 81606  
Lincoln, NE 68501-1606  
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

Prof. David Wells, Department of Mathematics  
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## 2006 AIME

The AIME will be held on Tuesday, March 7, 2006 with the alternate on Wednesday, March 22, 2006. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 18 and 19, 2006. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

## **PUBLICATIONS**

A complete listing of current publications, with ordering instructions, is at our web site:



2006

AMC 12 – Contest A

# DO NOT OPEN UNTIL TUESDAY, January 31, 2006

**\*\*Administration On An Earlier Date Will Disqualify  
Your School's Results\*\***

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE JANUARY 31.** Nothing is needed from inside this package until January 31.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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