# Tuesday, FEBRUARY 10, 2004

55th Annual American Mathematics Contest 12

# AMC 12

# Contest A

**The MATHEMATICAL ASSOCIATION OF AMERICA**
**American Mathematics Competitions**

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a twenty-five question, multiple choice test. Each question is followed by answers marked A,B,C,D and E. Only one of these is correct.
3. The answers to the problems are to be marked on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** working time to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

*Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 22nd annual American Invitational Mathematics Examination (AIME) on Tuesday, March 23, 2004 or Tuesday, April 6, 2004. More details about the AIME and other information are on the back page of this test booklet.*

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. Alicia earns \$20 per hour, of which 1.45% is deducted to pay local taxes. How many cents per hour of Alicia's wages are used to pay local taxes?

   **(A)** 0.0029 **(B)** 0.029 **(C)** 0.29 **(D)** 2.9 **(E)** 29

2. On the AMC 12, each correct answer is worth 6 points, each incorrect answer is worth 0 points, and each problem left unanswered is worth 2.5 points. If Charlyn leaves 8 of the 25 problems unanswered, how many of the remaining problems must she answer correctly in order to score at least 100?

   **(A)** 11 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 17

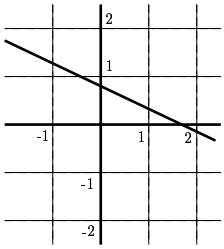
3. For how many ordered pairs of positive integers $(x, y)$ is $x + 2y = 100$?

   **(A)** 33 **(B)** 49 **(C)** 50 **(D)** 99 **(E)** 100

4. Bertha has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Bertha has a total of 30 daughters and granddaughters, and no great-granddaughters. How many of Bertha's daughters and granddaughters have no daughters?

   **(A)** 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

5. The graph of a line $y = mx + b$ is shown. Which of the following is true?




   **(A)** $mb < -1$ **(B)** $-1 < mb < 0$ **(C)** $mb = 0$
   **(D)** $0 < mb < 1$ **(E)** $mb > 1$

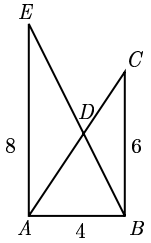
6. Let $U = 2 \cdot 2004^{2005}$, $V = 2004^{2005}$, $W = 2003 \cdot 2004^{2004}$, $X = 2 \cdot 2004^{2004}$, $Y = 2004^{2004}$ and $Z = 2004^{2003}$. Which of the following is largest?

   **(A)** $U - V$ **(B)** $V - W$ **(C)** $W - X$ **(D)** $X - Y$ **(E)** $Y - Z$

7. A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token into a discard pile. The game ends when some player runs out of tokens. Players $A$, $B$, and $C$ start with 15, 14, and 13 tokens, respectively. How many rounds will there be in the game?

   **(A)** 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

8. In the Figure, $\angle EAB$ and $\angle ABC$ are right angles, $AB = 4$, $BC = 6$, $AE = 8$, and $\overline{AC}$ and $\overline{BE}$ intersect at $D$. What is the difference between the areas of $\triangle ADE$ and $\triangle BDC$?




**(A)** 2 **(B)** 4 **(C)** 5 **(D)** 8 **(E)** 9

9. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

**(A)** 10 **(B)** 25 **(C)** 36 **(D)** 50 **(E)** 60

10. The sum of 49 consecutive integers is $7^5$. What is their median?

**(A)** 7 **(B)** $7^2$ **(C)** $7^3$ **(D)** $7^4$ **(E)** $7^5$

11. The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

**(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

12. Let $A = (0, 9)$ and $B = (0, 12)$. Points $A'$ and $B'$ are on the line $y = x$, and $\overline{AA'}$ and $\overline{BB'}$ intersect at $C = (2, 8)$. What is the length of $\overline{A'B'}$?

**(A)** 2 **(B)** $2\sqrt{2}$ **(C)** 3 **(D)** $2 + \sqrt{2}$ **(E)** $3\sqrt{2}$

13. Let $S$ be the set of points $(a, b)$ in the coordinate plane, where each of $a$ and $b$ may be $-1$, 0, or 1. How many distinct lines pass through at least two members of $S$?

**(A)** 8 **(B)** 20 **(C)** 24 **(D)** 27 **(E)** 36

14. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

**(A)** 1 **(B)** 4 **(C)** 36 **(D)** 49 **(E)** 81

15. Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

**(A)** 250 **(B)** 300 **(C)** 350 **(D)** 400 **(E)** 500

16. The set of all real numbers $x$ for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$

is defined is $\{x \mid x > c\}$. What is the value of $c$?

**(A)** 0 **(B)** $2001^{2002}$ **(C)** $2002^{2003}$ **(D)** $2003^{2004}$ **(E)** $2001^{2002^{2003}}$

17. Let $f$ be a function with the following properties:

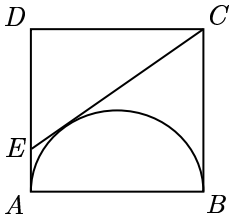
(i) $f(1) = 1$, and

(ii) $f(2n) = n \cdot f(n)$ for any positive integer $n$.

What is the value of $f(2^{100})$?

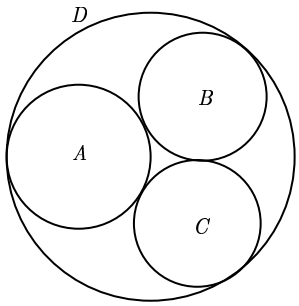
**(A)** 1 **(B)** $2^{99}$ **(C)** $2^{100}$ **(D)** $2^{4950}$ **(E)** $2^{9999}$

18. Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from $C$ intersects side $\overline{AD}$ at $E$. What is the length of $\overline{CE}$?




**(A)** $\frac{2+\sqrt{5}}{2}$ **(B)** $\sqrt{5}$ **(C)** $\sqrt{6}$ **(D)** $\frac{5}{2}$ **(E)** $5-\sqrt{5}$

19. Circles $A, B$, and $C$ are externally tangent to each other and internally tangent to circle $D$. Circles $B$ and $C$ are congruent. Circle $A$ has radius 1 and passes through the center of $D$. What is the radius of circle $B$?




**(A)** $\frac{2}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{1+\sqrt{3}}{3}$

20. Select numbers $a$ and $b$ between 0 and 1 independently and at random, and let $c$ be their sum. Let $A$, $B$, and $C$ be the results when $a$, $b$, and $c$, respectively, are rounded to the nearest integer. What is the probability that $A + B = C$?

**(A)** $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

21. If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

**(A)** $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{\sqrt{5}}{5}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{4}{5}$

22. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

**(A)** $3+\frac{\sqrt{30}}{2}$ **(B)** $3+\frac{\sqrt{69}}{3}$ **(C)** $3+\frac{\sqrt{123}}{4}$ **(D)** $\frac{52}{9}$ **(E)** $3+2\sqrt{2}$

23. A polynomial

$$P(x) = c_{2004}x^{2004} + c_{2003}x^{2003} + \cdots + c_1x + c_0$$

has real coefficients with $c_{2004} \neq 0$ and 2004 distinct complex zeros $z_k = a_k + b_k i$, $1 \leq k \leq 2004$ with $a_k$ and $b_k$ real, $a_1 = b_1 = 0$, and

$$\sum_{k=1}^{2004} a_k = \sum_{k=1}^{2004} b_k.$$

Which of the following quantities can be a nonzero number?

**(A)** $c_0$ **(B)** $c_{2003}$ **(C)** $b_2b_3\ldots b_{2004}$ **(D)** $\sum_{k=1}^{2004} a_k$ **(E)** $\sum_{k=1}^{2004} c_k$

24. A plane contains points $A$ and $B$ with $AB = 1$. Let $S$ be the union of all disks of radius 1 in the plane that cover $\overline{AB}$. What is the area of $S$?

**(A)** $2\pi + \sqrt{3}$ **(B)** $\frac{8\pi}{3}$ **(C)** $3\pi - \frac{\sqrt{3}}{2}$ **(D)** $\frac{10\pi}{3} - \sqrt{3}$ **(E)** $4\pi - 2\sqrt{3}$

25. For each integer $n \geq 4$, let $a_n$ denote the base-$n$ number $0.\overline{133}_n$. The product $a_4 a_5 \ldots a_{99}$ can be expressed as $\frac{m}{n!}$, where $m$ and $n$ are positive integers and $n$ is as small as possible. What is the value of $m$?

**(A)** 98 **(B)** 101 **(C)** 132 **(D)** 798 **(E)** 962

## WRITE TO US!

*Correspondence about the problems and solutions for this* ***AMC 12*** *should be addressed to:*

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068
Phone: 724/334-6749; Fax: 724/334-6110; email: dmw8@psu.edu

*Orders for any of the publications listed below should be addressed to:*

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## 2004 AIME

The AIME will be held on Tuesday, March 23, 2004 with the alternate on April 6, 2004. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 1% of the AMC 10 or receive a score of 100 or above on the AMC 12. Alternately, you must be in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) in late Spring. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

## PUBLICATIONS

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2004

AMC 12 - Contest A

# DO NOT OPEN UNTIL TUESDAY, FEBRUARY 10, 2004

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHER'S MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 10.** Nothing is needed from inside this package until February 10.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form A found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
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