The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

54th Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 11, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. (D) Each even counting number, beginning with 2, is one more than the preceding odd counting number. Therefore the difference is $(1)(2003) = 2003$.

2. (B) The cost for each member is the price of two pairs of socks, \$8, and two shirts, \$18, for a total of \$26. So there are $2366/26 = 91$ members.

3. (D) The total volume of the eight removed cubes is $8 \times 3^3 = 216$ cubic centimeters, and the volume of the original box is $15 \times 10 \times 8 = 1200$ cubic centimeters. Therefore the volume has been reduced by $\left(\frac{216}{1200}\right)(100\%) = 18\%$.

4. (A) Mary walks a total of 2 km in 40 minutes. Because 40 minutes is 2/3 hr, her average speed, in km/hr, is $2/(2/3) = 3$.

5. (E) Since the last two digits of $AMC10$ and $AMC12$ sum to 22, we have

$$AMC + AMC = 2(AMC) = 1234.$$

   Hence $AMC = 617$, so $A = 6$, $M = 1$, $C = 7$, and $A + M + C = 6 + 1 + 7 = 14$.

6. (C) For example, $-1 \heartsuit 0 = |-1-0| = 1 \neq -1$. All the other statements are true:
   (A) $x \heartsuit y = |x-y| = |-(y-x)| = |y-x| = y \heartsuit x$ for all x and y.
   (B) $2(x \heartsuit y) = 2|x-y| = |2x-2y| = (2x) \heartsuit (2y)$ for all x and y.
   (D) $x \heartsuit x = |x-x| = 0$ for all x.
   (E) $x \heartsuit y = |x-y| > 0$ if $x \neq y$.

7. (B) The longest side cannot be greater than 3, since otherwise the remaining two sides would not be long enough to form a triangle. The only possible triangles have side lengths 1–3–3 or 2–2–3.

8. (E) The factors of 60 are

$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, \text{ and } 60.$$

   Six of the twelve factors are less than 7, so the probability is 1/2.

9. (D) The set S is symmetric about the line $y = x$ and contains $(2, 3)$, so it must also contain $(3, 2)$. Also S is symmetric about the x-axis, so it must contain $(2, -3)$ and $(3, -2)$. Finally, since S is symmetric about the y-axis, it must contain $(-2, 3)$, $(-3, 2)$, $(-2, -3)$, and $(-3, -2)$. Since the resulting set of 8 points is symmetric about both coordinate axes, it is also symmetric about the origin.

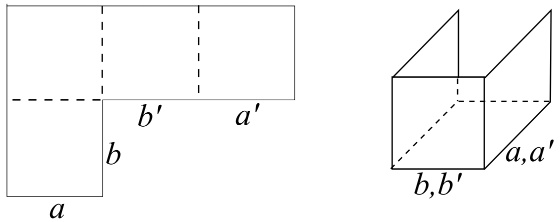
10. (D) Al, Bert, and Carl are to receive, respectively, $1/2, 1/3$, and $1/6$ of the candy. However, each believes he is the first to arrive. Therefore they leave behind, respectively, $1/2, 2/3$, and $5/6$ of the candy that was there when they arrived. The amount of unclaimed candy is $(1/2)(2/3)(5/6) = 5/18$ of the original amount, regardless of the order in which they arrive.

11. (C) Rescaling to different units does not affect the ratio of the areas, so let the perimeter be 12. Each side of the square then has length 3, and each side of the triangle has length 4. The diameter of the circle circumscribing the square is the diagonal of the square, $3\sqrt{2}$. Thus $A = \pi(3\sqrt{2}/2)^2 = 9\pi/2$. The altitude of the triangle is $2\sqrt{3}$, so the radius of the circle circumscribing the triangle is $4\sqrt{3}/3$, and $B = \pi(4\sqrt{3}/3)^2 = 16\pi/3$. Therefore

$$\frac{A}{B} = \frac{9\pi}{2}\frac{3}{16\pi} = \frac{27}{32}.$$

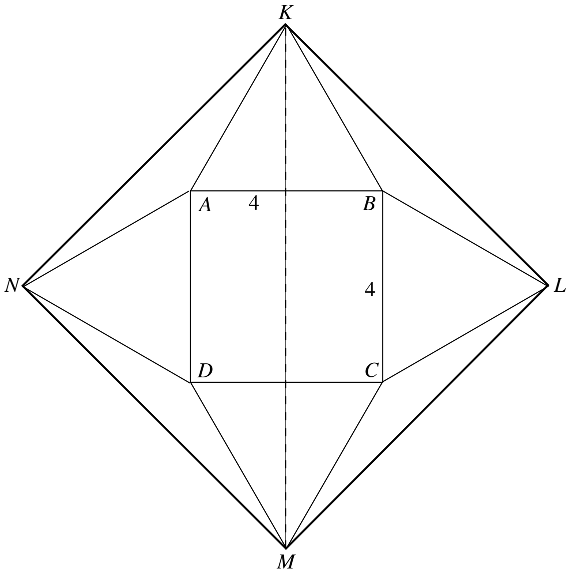
12. (E) Let R1, ..., R5 and B3, ..., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, ..., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, ..., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

13. (E) If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached at any of the six remaining edges.



14. (D) Quadrilateral $KLMN$ is a square because it has 90° rotational symmetry, which implies that each pair of adjacent sides is congruent and perpendicular. Since $ABCD$ has sides of length 4 and K is $2\sqrt{3}$ from side $\overline{AB}$, the length of the diagonal $\overline{KM}$ is $4 + 4\sqrt{3}$. Thus the area is

$$\frac{1}{2}(4 + 4\sqrt{3})^2 = 32 + 16\sqrt{3}.$$



OR

Note that $m(\angle NAK) = 150^\circ$. By the Law of Cosines,

$$(NK)^2 = 4^2 + 4^2 - 2(4)(4)\left(-\frac{\sqrt{3}}{2}\right) = 32 + 16\sqrt{3}.$$

Since $KLMN$ is a square, its area is $(NK)^2 = 32 + 16\sqrt{3}$.

15. (C) First note that the area of the region determined by the triangle topped by the semicircle of diameter 1 is

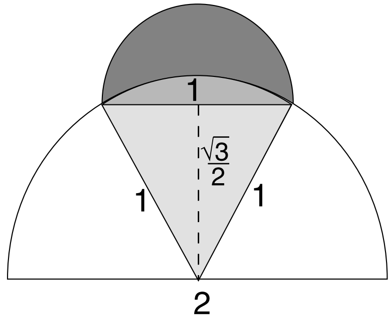
$$\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2 = \frac{1}{6}\pi.$$

So the area of the lune is

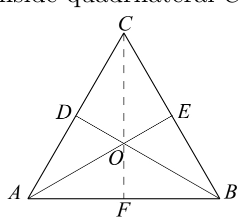
$$\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$



Note that the answer does not depend on the position of the lune on the semicircle.

16. (C) Since the three triangles ABP, ACP, and BCP have equal bases, their areas are proportional to the lengths of their altitudes.

Let O be the centroid of $\triangle ABC$, and draw medians $\overline{AOE}$ and $\overline{BOD}$. Any point above $\overline{BOD}$ will be farther from $\overline{AB}$ than from $\overline{BC}$, and any point above $\overline{AOE}$ will be farther from $\overline{AB}$ than from $\overline{AC}$. Therefore the condition of the problem is met if and only if P is inside quadrilateral $CDOE$.



If $\overline{CO}$ is extended to F on $\overline{AB}$, then $\triangle ABC$ is divided into six congruent triangles, of which two comprise quadrilateral $CDOE$. Thus $CDOE$ has one-third the area of $\triangle ABC$, so the required probability is $1/3$.

OR

By symmetry, each of $\triangle ABP$, $\triangle ACP$, and $\triangle BCP$ is largest with the same probability, so the probability must be $1/3$ for each.

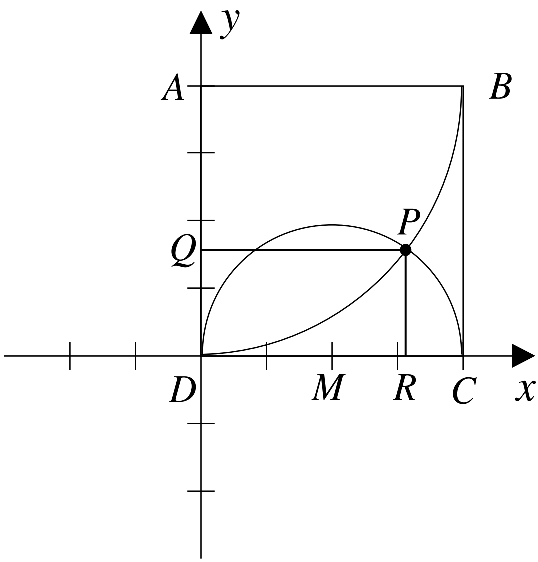
17. (B) Place an xy-coordinate system with origin at D and points C and A on the positive x- and y-axes, respectively. Then the circle centered at M has equation

$$(x-2)^2 + y^2 = 4,$$

and the circle centered at A has equation

$$x^2 + (y-4)^2 = 16.$$

Solving these equations for the coordinates of P gives $x = 16/5$ and $y = 8/5$, so the answer is $16/5$.



OR

We have $AP = AD = 4$ and $PM = MD = 2$, so $\triangle ADM$ is congruent to $\triangle APM$, and $\angle APM$ is a right angle. Draw $\overline{PQ}$ and $\overline{PR}$ perpendicular to $\overline{AD}$ and $\overline{CD}$, respectively. Note that $\angle APQ$ and $\angle MPR$ are both complements of $\angle QPM$. Thus $\triangle APQ$ is similar to $\triangle MPR$, and

$$\frac{AQ}{MR} = \frac{AP}{MP} = \frac{4}{2} = 2.$$

Let $MR = x$. Then $AQ = 2x$, $PR = QD = 4 - 2x$, and $PQ = RD = x + 2$. Therefore

$$2 = \frac{AQ}{MR} = \frac{PQ}{PR} = \frac{x+2}{4-2x},$$

so $x = 6/5$ and $PQ = 6/5 + 2 = 16/5$.

OR

Let $\angle MAD = \alpha$. Then

$$PQ = (PA)\sin(\angle PAQ) = 4\sin(2\alpha) = 8\sin\alpha\cos\alpha = 8\left(\frac{2}{\sqrt{20}}\right)\left(\frac{4}{\sqrt{20}}\right) = \frac{16}{5}.$$

18. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \le n \le 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

19. (D) The original parabola has equation $y = a(x-h)^2 + k$, for some a, h, and k, with $a \neq 0$. The reflected parabola has equation $y = -a(x-h)^2 - k$. The translated parabolas have equations

$$f(x) = a(x - h \pm 5)^2 + k \quad \text{and} \quad g(x) = -a(x - h \mp 5)^2 - k,$$

so

$$(f+g)(x) = \pm 20a(x-h).$$

Since $a \neq 0$, the graph is a non-horizontal line.

20. (A) Since the first group of five letters contains no A's, it must contain k B's and $(5-k)$ C's for some integer k with $0 \leq k \leq 5$. Since the third group of five letters contains no C's, the remaining k C's must be in the second group, along with $(5-k)$ A's.

Similarly, the third group of five letters must contain k A's and $(5-k)$ B's. Thus each arrangement that satisfies the conditions is determined uniquely by the location of the k B's in the first group, the k C's in the second group, and the k A's in the third group.

For each k, the letters can be arranged in $\binom{5}{k}^3$ ways, so the total number of arrangements is

$$\sum_{k=0}^{5} \binom{5}{k}^3.$$

21. (D) Since $P(0) = 0$, we have $e = 0$ and $P(x) = x(x^4 + ax^3 + bx^2 + cx + d)$. Suppose that the four remaining x-intercepts are at p, q, r, and s. Then

$$x^4 + ax^3 + bx^2 + cx + d = (x-p)(x-q)(x-r)(x-s),$$

and $d = pqrs \neq 0$.
Any of the other constants could be zero. For example, consider

$$P_1(x) = x^5 - 5x^3 + 4x = x(x+2)(x+1)(x-1)(x-2)$$

and

$$P_2(x) = x^5 - 5x^4 + 20x^2 - 16x = x(x+2)(x-1)(x-2)(x-4).$$

OR

Since $P(0) = 0$, we must have $e = 0$, so

$$P(x) = x(x^4 + ax^3 + bx^2 + cx + d).$$

If $d = 0$, then

$$P(x) = x(x^4 + ax^3 + bx^2 + cx) = x^2(x^3 + ax^2 + bx + c),$$

which has a double root at $x = 0$. Hence $d \neq 0$.

OR

There is also a calculus-based solution. Since $P(x)$ has five distinct zeros and $x = 0$ is one of the zeros, it must be a zero of multiplicity one. This is equivalent to having $P(0) = 0$, but $P'(0) \neq 0$. Since

$$P'(x) = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d, \quad \text{we must have} \quad 0 \neq P'(0) = d.$$

22. (C) Since there are twelve steps between $(0, 0)$ and $(5, 7)$, A and B can meet only after they have each moved six steps. The possible meeting places are $P_0 = (0, 6)$, $P_1 = (1, 5)$, $P_2 = (2, 4)$, $P_3 = (3, 3)$, $P_4 = (4, 2)$, and $P_5 = (5, 1)$. Let a_i and b_i denote the number of paths to P_i from $(0, 0)$ and $(5, 7)$, respectively. Since A has to take i steps to the right and B has to take $i + 1$ steps down, the number of ways in which A and B can meet at P_i is

$$a_i \cdot b_i = \binom{6}{i}\binom{6}{i+1}.$$

Since A and B can each take 2^6 paths in six steps, the probability that they meet is

$$\begin{aligned}\sum_{i=0}^{5}\left(\frac{a_i}{2^6}\right)\left(\frac{b_i}{2^6}\right) =& \frac{\binom{6}{0}\binom{6}{1} + \binom{6}{1}\binom{6}{2} + \binom{6}{2}\binom{6}{3} + \binom{6}{3}\binom{6}{4} + \binom{6}{4}\binom{6}{5} + \binom{6}{5}\binom{6}{6}}{2^{12}} \\ =& \frac{99}{512} \approx 0.20.\end{aligned}$$

OR

Consider the $\binom{12}{5}$ walks that start at $(0, 0)$, end at $(5, 7)$, and consist of 12 steps, each one either up or to the right. There is a one-to-one correspondence between these walks and the set of (A, B)-paths where A and B meet. In particular, given one of the $\binom{12}{5}$ walks from $(0, 0)$ to $(5, 7)$, the path followed by A consists of the the first six steps of the walk, and the path followed by B is obtained by starting at $(5, 7)$ and reversing the last six steps of the walk. There are 2^6 paths that take 6 steps from $(0, 0)$ and 2^6 paths that take 6 steps from $(5, 7)$, so there are 2^{12} pairs of paths that A and B can take. The probability that they meet is

$$P = \frac{1}{2^{12}}\binom{12}{5} = \frac{99}{2^9}.$$

23. (B) We have

$$\begin{aligned}1! \cdot 2! \cdot 3! \cdots 9! =& (1)(1 \cdot 2)(1 \cdot 2 \cdot 3) \cdots (1 \cdot 2 \cdots 9) \\ =& 1^9 2^8 3^7 4^6 5^5 6^4 7^3 8^2 9^1 = 2^{30} 3^{13} 5^5 7^3.\end{aligned}$$

The perfect square divisors of that product are the numbers of the form

$$2^{2a}3^{2b}5^{2c}7^{2d}$$

with $0 \le a \le 15$, $0 \le b \le 6$, $0 \le c \le 2$, and $0 \le d \le 1$. Thus there are $(16)(7)(3)(2) = 672$ such numbers.

24. (B) We have

$$\begin{aligned}\log_a \frac{a}{b} + \log_b \frac{b}{a} &= \log_a a - \log_a b + \log_b b - \log_b a\\ &= 1 - \log_a b + 1 - \log_b a\\ &= 2 - \log_a b - \log_b a.\end{aligned}$$

Let $c = \log_a b$, and note that $c > 0$ since a and b are both greater than 1. Thus

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c} = \frac{c^2 - 2c + 1}{-c} = \frac{(c-1)^2}{-c} \le 0.$$

This expression is 0 when $c = 1$, that is, when $a = b$.

OR

As above

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c}$$

From the Arithmetic-Geometric Mean Inequality we have

$$\frac{c + 1/c}{2} \ge \sqrt{c \cdot \frac{1}{c}} = 1, \quad \text{so} \quad c + \frac{1}{c} \ge 2$$

and

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - \left(c + \frac{1}{c}\right) \le 0$$

with equality when $c = \frac{1}{c}$, that is, when $a = b$.

25. (C) The domain of f is $\{x \mid ax^2 + bx \ge 0\}$. If $a = 0$, then for every positive value of b, the domain and range of f are each equal to the interval $[0, \infty)$, so 0 is a possible value of a.
If $a \ne 0$, the graph of $y = ax^2 + bx$ is a parabola with x-intercepts at $x = 0$ and $x = -b/a$. If $a > 0$, the domain of f is $(-\infty, -b/a] \cup [0, \infty)$, but the range of f cannot contain negative numbers. If $a < 0$, the domain of f is $[0, -b/a]$. The maximum value of f occurs halfway between the x-intercepts, at $x = -b/2a$, and

$$f\left(-\frac{b}{2a}\right) = \sqrt{a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right)} = \frac{b}{2\sqrt{-a}}.$$

Hence, the range of f is $[0, b/2\sqrt{-a}]$. For the domain and range to be equal, we must have

$$-\frac{b}{a} = \frac{b}{2\sqrt{-a}} \quad \text{so} \quad 2\sqrt{-a} = -a.$$

The only solution is $a = -4$. Thus there are two possible values of a, and they are $a = 0$ and $a = -4$.

The

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