AMERICAN MATHEMATICS COMPETITIONS
1st Annual Mathematics Contest →10

AMC → 10
Solutions Pamphlet
TUESDAY, FEBRUARY 15, 2000

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This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC →10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, eMail, the Web or media of any type is a violation of the copyright law.

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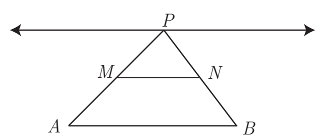
1. **Answer (E):** Factor 2001 into primes to get $2001 = 3 \cdot 23 \cdot 29$. The largest possible sum of three distinct factors whose product is the one which combines the two largest prime factors, namely $I = 23 \cdot 29 = 667$, $M = 3$, and $O = 1$, so the largest possible sum is $1 + 3 + 667 = 671$.

2. **Answer (A):** $2000(2000^{2000}) = (2000^1)(2000^{2000}) = 2000^{1+2000} = 2000^{2001}$. All the other options are greater than 2000^{2001}.

3. **Answer (B):** Since Jenny ate 20% of the jellybeans remaining each day, 80% of the jellybeans are left at the end of each day. If x is the number of jellybeans in the jar originally, then$(0.8)^2x = 32$. Thus $x = 50$.

4. **Answer (D):** Since Chandra paid extra \$5.06 in January, her December connect time must have cost her \$5.06. Therefore, her monthly fee is \$12.48 − \$5.06 = \$7.42.

5. **Answer (B):** Since $\triangle ABP$ is similar to $\triangle MNP$ and $PM = \frac{1}{2} \cdot AP$, it follows that $MN = \frac{1}{2} \cdot AB$. Since the base AB and the altitude to AB of $\triangle ABP$ do not change, the area does not change. The altitude of the trapezoid is half that of the triangle, and the bases do not change as P changes, so the area of the trapezoid does not change. Only the perimeter changes (reaching a minimum when $\triangle ABP$ is isosceles).

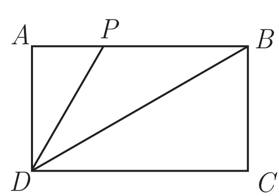


6. **Answer (C):** The sequence of units digits is

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots$$

The digit 6 is the last of the ten digits to appear.

7. **Answer (B):** Both triangles APD and CBD are $30-60-90^\circ$ triangles. Thus $DP = \frac{2\sqrt{3}}{3}$ and $DB = 2$. Since $\angle BDP = \angle PDB$, it follows that $PB = PD = \frac{2\sqrt{3}}{3}$. Hence the perimeter of $\triangle BDP$ is $\frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} + 2 = 2 + \frac{4\sqrt{3}}{3}$.



8. **Answer (D):** Let f and s represent the numbers of freshmen and sophomores at the school, respectively. According to the given condition, $(2/5)f = (4/5)s$. Thus, $f = 2s$. That is, there are twice as many freshmen as sophomores.

9. **Answer (C):** Since $x < 2$, it follows that $|x-2| = 2-x$. If $2-x = p$, then $x = 2-p$. Thus $x - p = 2 - 2p$.

10. **Answer (D):** By the *Triangle Inequality*, each of x and y can be any number strictly between 2 and 10, so $0 \le |x-y| < 8$. Therefore, the smallest positive number that is not a possible value of $|x-y|$ is $10-2 = 8$.

11. **Answer (C):** There are five prime numbers between 4 and 18: 5,7,11,13, and 17. Hence the product of any two of these is odd and the sum is even. Because $xy-(x+y) = (x-1)(y-1)-1$ increases as either x or y increases (since both x and y are bigger than 1), the answer must be an odd number that is no smaller than $23 = 5\cdot 7 - (5+7)$ and no larger than $191 = 13\cdot 17 - (13+17)$. The only possibility among the options is 119, and indeed $119 = 11\cdot 13 - (11+13)$.

12. **Answer (C):** Calculating the number of squares in the first few figures uncovers a pattern. Figure 0 has $2(0)+1 = 2(0^2)+1$ squares, figure 1 has $2(1)+3 = 2(1^2)+3$ squares, figure 2 has $2(1+3)+5 = 2(2^2)+5$ squares, and figure 3 has $2(1+3+5)+7 = 2(3^2)+7$ squares. In general, the number of unit squares in figure n is

$$2(1+3+5+\cdots+(2n-1)) + 2n + 1 = 2(n^2) + 2n + 1.$$

Therefore, the figure 100 has $2(100^2) + 2\cdot 100 + 1 = 20201$.

OR

Each figure can be considered as a large square with identical small pieces deleted from each of the four corners. Figure 1 has $3^2 - 4(1)$ unit squares, figure 2 has $5^2 - 4(1+2)$ unit squares, and figure 3 has $7^2 - 4(1+2+3)$ unit squares. In general, figure n has

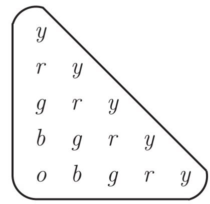
$$(2n-1)^2 - 4(1+2+\cdots+n) = (2n+1)^2 - 2n(n+1) \text{ unit squares.}$$

Thus figure 100 has $201^2 - 200(101) = 20201$ unit squares.

OR

The number of unit squares in figure n is the sum of the first n positive odd integers plus the sum of the first $n+1$ positive odd integers. Since the sum of the first k positive odd integers is k^2, figure n has $n^2 + (n+1)^2$ unit squares. So figure 100 has $100^2 + 101^2 = 20201$ unit squares.

13. **Answer (B):** To avoid having two yellow pegs in the same row or column, there must be exactly one yellow peg in each row and in each column. Hence, starting at the top of the array, the peg in the first row must be yellow, the second peg of the second row must be yellow, the third peg of the third row must be yellow, etc. To avoid having two red pegs in some row, there must be a red peg in each of rows 2,3,4, and 5. The red pegs must be in the first position of the second row, the second position of the third row, etc. Continuation yields exactly one ordering that meets the requirements, as shown.



14. **Answer (C):** Note that the integer average condition means that the sum of the scores of the first n students is a multiple of n. The scores of the first two students must be both even or both odd, and the sum of the scores of the first three students must be divisible by 3. The remainders when 71, 76, 80, 82, and 91 are divided by 3 are 2, 1, 2, 1, and 1, respectively. Thus the only sum of three scores divisible by 3 is $76 + 82 + 91 = 249$, so the first two scores entered are 76 and 82 (in some order), and the third score is 91. Since 249 is 1 larger than a multiple of 4, the fourth score must be 3 larger than a multiple of 4, and the only possible is 71, leaving 80 as the score of the fifth student.

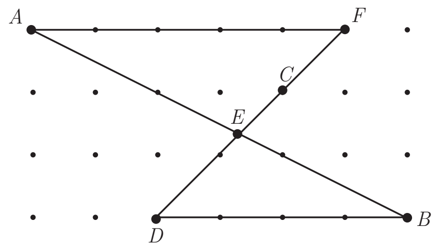
15. **Answer (E):** Find the common denominator and replace the ab in the numerator with $a - b$ to get

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2 - (ab)^2}{ab} \\ &= \frac{a^2 + b^2 - (a-b)^2}{ab} \\ &= \frac{a^2 + b^2 - (a^2 - 2ab + b^2)}{ab} \\ &= \frac{2ab}{ab} = 2.\end{aligned}$$

OR

Note that $a = a/b - 1$ and $b = 1 - b/a$. It follows that $\frac{a}{b} + \frac{b}{a} - ab = (a+1) + (1-b) - (a-b) = 2$.

16. **Answer (B):** Extend $\overline{DC}$ to F. Triangle FAE and DBE are similar with ratio $5:4$. Thus $AE = 5 \cdot AB/9$, $AB = \sqrt{3^2+6^2} = \sqrt{45} = 3\sqrt{5}$, and $AE = 5(3\sqrt{5})/9 = 5\sqrt{5}/3$.

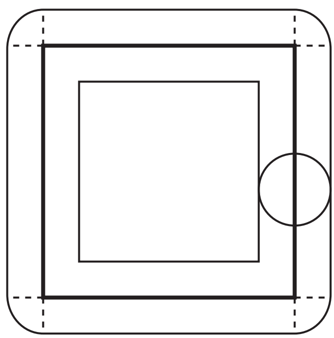


OR

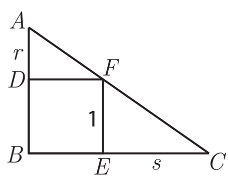
Coordinatize the points so that $A = (0,3)$, $B = (6,0)$, $C = (4,2)$, and $D = (2,0)$. Then the line through A and B is given by $x + 2y = 6$, and the line through C and D is given by $x - y = 2$. Solve these simultaneously to get $E = \left(\frac{10}{3}, \frac{4}{3}\right)$. Hence $AE = \sqrt{(\frac{10}{3}-0)^2 + (\frac{4}{3}-3)^2} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$.

17. **Answer (D):** Neither of the exchanges *quarter* → *five nickels* nor *nickel* → *five pennies* changes the total value of Boris's coins. The exchange *penny* → *five quarters* increase the total value of Boris's coins by \$1.24. Hence, Boris must have \$.01 + \$1.24n after n uses of the last exchange. Only option D is of this form: $745 = 1 + 124 \cdot 6$. In cents, option A is 115 more than a multiple of 124, B is 17 more than a multiple of 124, C is 10 more than a multiple of 124, and E is 39 more than a multiple of 124.

18. **Answer (C):** At any point on Charlyn's walk, she can see all the points inside a circle of radius 1 km. The portion of the viewable region inside the square consists of the interior of the square except for a smaller square with side length 3 km. This portion of the viewable region has area $(25 - 9)\text{ km}^2$. The portion of the viewable region outside the square consists of four rectangles, each 5 km by 1 km, and four quarter-circles, each with a radius of 1 km. This portion of the viewable region has area $4(5 + \frac{\pi}{4}) = (20 + \pi)\text{ km}^2$. The area of the entire viewable region is $36 + \pi \approx 30\text{ km}^2$.

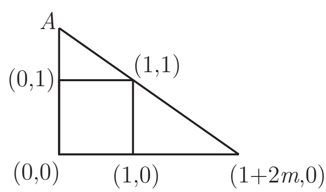


19. **Answer (D):** With out loss of generality, let the side of the square have length 1 unit and let the area of triangle ADF be m. Let $AD = r$ and $EC = s$. Because triangles ADF and FEC are similar, $s/1 = 1/r$. Since $\frac{1}{2}r = m$, the area of triangle FEC is $\frac{1}{2}s = \frac{1}{2r} = \frac{1}{4m}$.



OR

Let $B = (0,0)$, $E = (1,0)$, $F = (1,1)$ and $D = (0,1)$ be the vertices of the square. Let $C = (1 + 2m, 0)$, and notice that the area of $BEFD$ is 1 and the area of triangle FEC is m. The slope of the line through C and F is $-\frac{1}{2m}$; thus, it intersects the y-axis at $A = \left(0, 1 + \frac{1}{2m}\right)$. The area of triangle ADF is therefore $\frac{1}{4m}$.

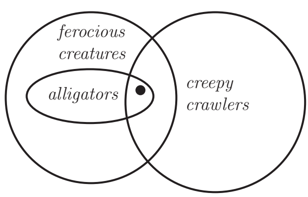


20. **Answer (C):** Note that

$$AMC+AM+MC+CA=(A+1)(M+1)(C+1)-(A+M+C)-1=pqr-11,$$

where p, q, and r are positive integers whose sum is 13. A case-by-case analysis shows that pqt is largest when two of the numbers p, q, r are 4 and the third is 5. Thus the answer is $4\cdot 4\cdot 5-11=69$.

21. **Answer (B):** From the conditions we can conclude that some creepy crawlers are ferocious (since some are alligators). Hence, there are some ferocious creatures that are creepy crawlers, and thus II must be true. The diagram below shows that the only conclusion that can be drawn is existence of an animal in the region with the dot. Thus, neither I nor III follows from the given conditions.



22. **Answer (C):** Suppose that the whole family drank x cups of milk and y cups of coffee. Let n denote the number of people in the family. The information given implies that $x/4 + y/6 = (x + y)/n$. This leads to

$$3x(n - 4) = 2y(6 - n).$$

Since x and y are positive, the only positive integer n for which both sides have the same sign is $n = 5$.

OR

If Angela drank c cups of coffee and m cups of mile, then $0 < c < 1$ and $m + c = 1$. The number of people in the family is $6c + 4m = 4 + 2c$, which is an integer if and only if $c = \frac{1}{2}$. Thus, there are 5 people in the family.

23. **Answer (E):** If x were less than or equal to 2, then 2 would be both the median and the mode of the list. Thus $x > 2$. Consider the two cases $2 < x < 4$, and $x \geq 4$.
Case 1: If $2 < x < 4$, then 2 is the mode, x is the median, and $\frac{25+x}{7}$ is the mean, which must equal $2 - (x - 2)$, $\frac{x+2}{2}$, or $x + (x - 2)$, depending on the size of the mean relative to 2 and x. These give $x = \frac{3}{8}$, $x = \frac{36}{5}$, and $x = 3$, of which $x = 3$ is the only value between 2 and 4.
Case 2: If $x \geq 4$, then 4 is the median, 2 is the mode, and $\frac{25+x}{7}$ is the mean, which must be 0,3, or 6. Thus $x = -25$, -4, or 17, of which 17 is the only one of these values greater than or equal to 4.
Thus the x-value sum to $3 + 17 = 20$.

24. **Answer (B):** Let $x = 9z$. Then $f(3z) = f(9z/3) = f(3z) = (9z)^2 + 9z + 1 = 7$. Simplifying and solving the equation for z yields $81z^2 + 9z - 6 = 0$, so $3(3z + 1)(9z - 2) = 0$. Thus $z = -1/3$ or $z = 2/9$. The sum of these values is $-1/9$.
Note. The answer can also be obtained by using the sum-of-roots formula on $81z^2 + 9z - 6 = 0$. The sum of the roots is $-9/81 = -1/9$.

25. **Answer (A):** Note that, if a Tuesday is d days after a Tuesday, then d is a multiple of 7. Next, we need to consider whether any of the years $N - 1$, N, $N + 1$ is a leap year. If N is not a leap year, the 200^{th} day of year $N + 1$ is $365 - 300 + 200 = 265$ days after a Tuesday, and thus is a Monday, since 265 if 6 larger than a multiple of 7. Thus, year N is a leap year and the 200^{th} day of year $N + 1$ is another Tuesday (as given), being 266 days after a Tuesday. It follows that year $N - 1$ is not a leap year. Therefore, the 100^{th} day of year $N - 1$ precedes the given Tuesday in year N by $365 - 100 + 300 = 565$ days, and therefore is a Thursday, since $565 = 7 \cdot 80 + 5$ is 5 larger than a multiple of 7.